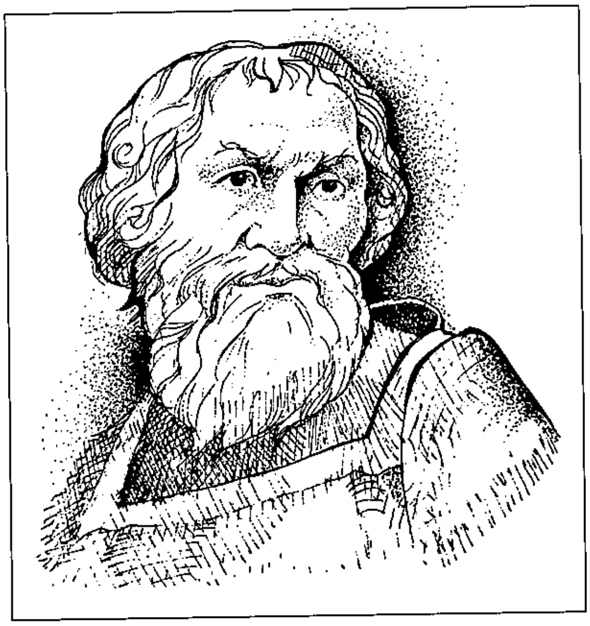
3 MENSURATION

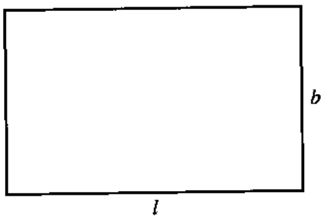


Archimides of Samos (287–212 B.C.) studied at Alexandria as a young man. One of the first to apply scientific thinking to everyday problems, he was a practical man of common sense. He gave proofs for finding the area, the volume and the centre of gravity of circles, spheres, conics and spirals. By drawing polygons with many sides, he arrived at a value of π between $3\frac{10}{71}$ and $3\frac{10}{70}$. He was killed in the siege of Syracuse at the age of 75.

31 Carry out calculations involving the perimeter and area of a rectangle and triangle, the circumference and area of a circle, the area of a parallelogram and trapezium, the volume of a cuboid, prism and cylinder, and the surface area of a cylinder.
Solve problems involving the arc length and sector area of a circle, the surface area and volume of a sphere, pyramid and cone

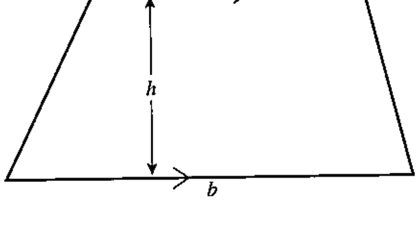
3.1 Area

Rectangle



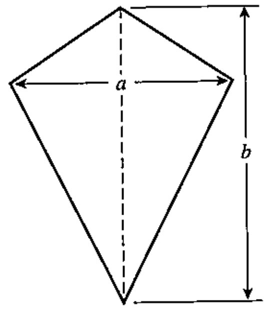
area $= l \times b$

Trapezium



area $= \frac{1}{2}(a + b)h$

Kite



area $= \frac{1}{2}a \times b$
$= \frac{1}{2} \times$ (product of diagonals)

Exercise 1

For questions 1 to 7, find the area of each shape. Decide which information to use: you may not need all of it.

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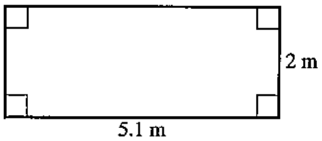

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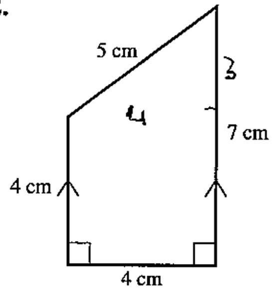

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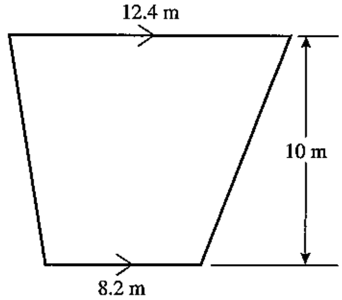

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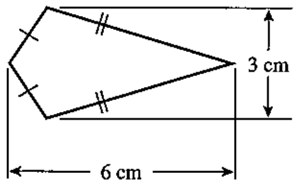

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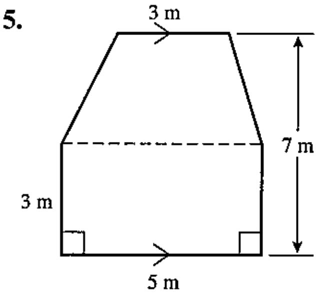

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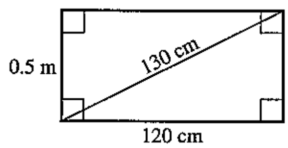

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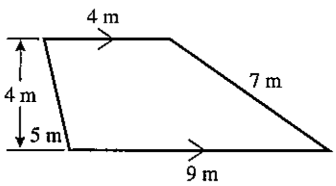
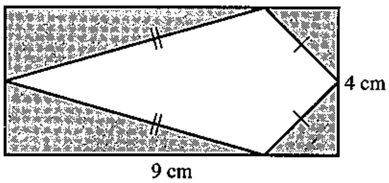
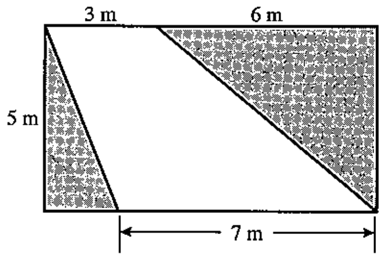


8. Find the area shaded.



9. Find the area shaded.



10. A rectangle has an area of $117\,m^2$ and a width of 9 m. Find its length.

11. A trapezium of area $105\,cm^2$ has parallel sides of length 5 cm and 9 cm. How far apart are the parallel sides?

12. A kite of area $252\,m^2$ has one diagonal of length 9 m. Find the length of the other diagonal.

13. A kite of area $40\,m^2$ has one diagonal 2 m longer than the other. Find the lengths of the diagonals.

14. A trapezium of area $140\,cm^2$ has parallel sides 10 cm apart and one of these sides is 16 cm long. Find the length of the other parallel side.

15. A floor 5 m by 20 m is covered by square tiles of side 20 cm. How many tiles are needed?

16. On squared paper draw the triangle with vertices at (1, 1), (5, 3), (3, 5). Find the area of the triangle.

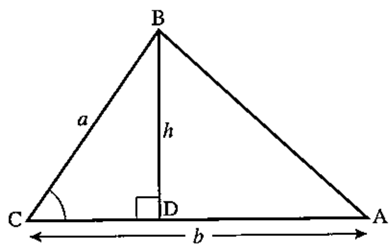
17. Draw the quadrilateral with vertices at (1, 1), (6, 2), (5, 5), (3, 6). Find the area of the quadrilateral.

18. A square wall is covered with square tiles. There are 85 tiles altogether along the two diagonals. How many tiles are there on the whole wall?

19. On squared paper draw a 7 × 7 square. Divide it up into nine smaller squares.

20. A rectangular field, 400 m long, has an area of 6 hectares. Calculate the perimeter of the field [1 hectare = $10\,000\,\text{m}^2$].

Triangle



area $= \frac{1}{2} \times b \times h$

In triangle BCD, $\sin C = \frac{h}{a}$

$\therefore \quad h = a \sin C$

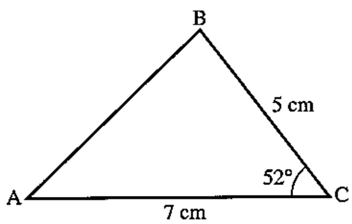
$\therefore \quad$ area of triangle $= \frac{1}{2} \times b \times a \sin C$

This formula is useful when *two sides* and the *included* angle are known.

You can find out more about sin θ in Trigonometry on page 183.

Example

Find the area of the triangle shown.

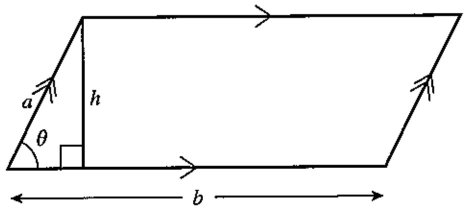


Area $= \frac{1}{2}ab \sin C$

$= \frac{1}{2} \times 5 \times 7 \times \sin 52°$

$= 13{\cdot}8\,\text{cm}^2$ (3 sig. fig.)

Parallelogram



area $= b \times h$

area $= ba \sin \theta$

Exercise 2

In questions **1** to **12** find the area of $\triangle ABC$ where $AB = c$, $AC = b$ and $BC = a$. (Sketch the triangle in each case.) You will need some basic trigonometry (see page 182).

1. $a = 7\,\text{cm}$, $b = 14\,\text{cm}$, $\hat{C} = 80°$.

2. $b = 11\,\text{cm}$, $a = 9\,\text{cm}$, $\hat{C} = 35°$.

3. $c = 12\,\text{m}$, $b = 12\,\text{m}$, $\hat{A} = 67{\cdot}2°$.

4. $a = 5\,\text{cm}$, $c = 6\,\text{cm}$, $\hat{B} = 11{\cdot}8°$.

5. $b = 4{\cdot}2\,\text{cm}$, $a = 10\,\text{cm}$, $\hat{C} = 120°$.

6. $a = 5\,\text{cm}$, $c = 8\,\text{cm}$, $\hat{B} = 142°$.

7. $b = 3{\cdot}2\,\text{cm}$, $c = 1{\cdot}8\,\text{cm}$, $\hat{B} = 10°$, $\hat{C} = 65°$.

8. $a = 7\,\text{m}$, $b = 14\,\text{m}$, $\hat{A} = 32°$, $\hat{B} = 100°$.

9. $a = b = c = 12\,\text{m}$.

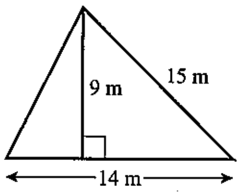
10. $a = c = 8\,\text{m}$, $\hat{B} = 72°$.

11. $b = c = 10\,\text{cm}$, $\hat{B} = 32°$.

12. $a = b = c = 0{\cdot}8\,\text{m}$.

In questions **13** to **20**, find the area of each shape.

13.



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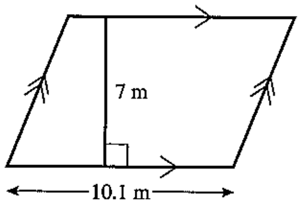

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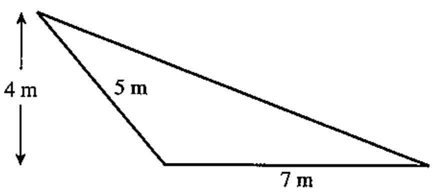

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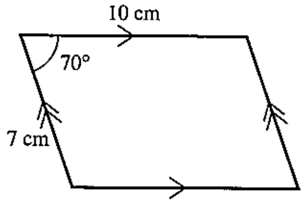

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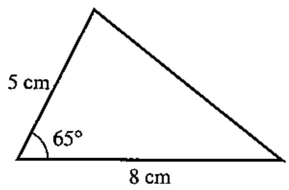

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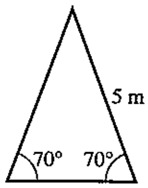
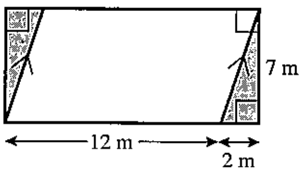


19. Find the area shaded.



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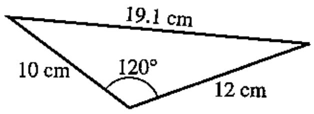
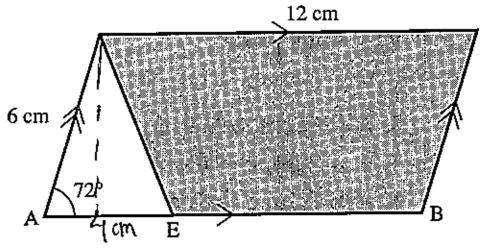


21. Find the area of a parallelogram ABCD with $AB = 7\,\text{m}$, $AD = 20\,\text{m}$ and $B\hat{A}D = 62°$.

22. Find the area of a parallelogram ABCD with $AD = 7\,\text{m}$, $CD = 11\,\text{m}$ and $B\hat{A}D = 65°$.

23. In the diagram if AE = $\frac{1}{3}$AB, find the area shaded.



24. The area of an equilateral triangle ABC is $50\,cm^2$. Find AB.

25. The area of a triangle ABC is $64\,cm^2$. Given AB = 11 cm and BC = 15 cm, find $A\hat{B}C$.

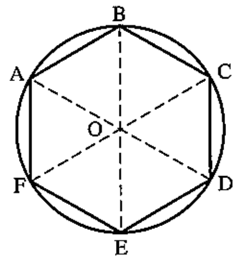
26. The area of a triangle XYZ is $11\,m^2$. Given YZ = 7 m and $X\hat{Y}Z = 130°$, find XY.

27. Find the length of a side of an equilateral triangle of area $10{\cdot}2\,m^2$.

28. A rhombus has an area of $40\,cm^2$ and adjacent angles of 50° and 130°. Find the length of a side of the rhombus.

You can find out more about special shapes and their properties in Unit 4.

29. A regular hexagon is circumscribed by a circle of radius 3 cm with centre O.



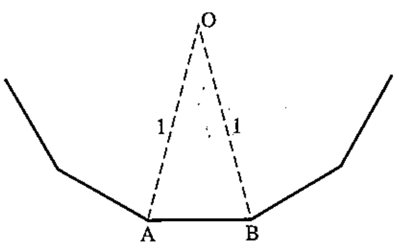
(a) What is angle EOD?

(b) Find the area of triangle EOD and hence find the area of the hexagon ABCDEF.

30. Hexagonal tiles of side 20 cm are used to tile a room which measures 6·25 m by 4·85 m. Assuming we complete the edges by cutting up tiles, how many tiles are needed?

31. Find the area of a regular pentagon of side 8 cm.

32. The diagram shows a part of the perimeter of a regular polygon with n sides



The centre of the polygon is at O and OA = OB = 1 unit.
(a) What is the angle AOB in terms of n?
(b) Work out an expression in terms of n for the area of the polygon.
(c) Find the area of polygons where $n = 6, 10, 300, 1000, 10\,000$. What do you notice?

33 The area of a regular pentagon is $600\,\text{cm}^2$. Calculate the length of one side of the pentagon.

3.2 The circle

For any circle, the ratio $\left(\frac{\text{circumference}}{\text{diameter}}\right)$ is equal to π.

The value of π is usually taken to be 3·14, but this is not an exact value. Through the centuries, mathematicians have been trying to obtain a better value for π.

For example, in the third century A.D., the Chinese mathematician Liu Hui obtained the value 3·14159 by considering a regular polygon having 3072 sides! Ludolph van Ceulen (1540–1610) worked even harder to produce a value correct to 35 significant figures. He was so proud of his work that he had this value of π engraved on his tombstone.

Electronic computers are now able to calculate the value of π to many thousands of figures, but its value is still not exact. It was shown in 1761 that π is an *irrational number* which, like $\sqrt{2}$ or $\sqrt{3}$ cannot be expressed exactly as a fraction.

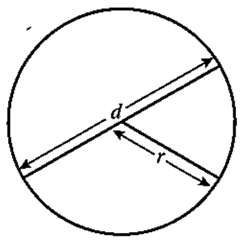
The first fifteen significant figures of π can be remembered from the number of letters in each word of the following sentence.

How I need a drink, cherryade of course, after the silly lectures involving Italian kangaroos.

There remain a lot of unanswered questions concerning π, and many mathematicians today are still working on them.

The following formulae should be memorised.

circumference $= \pi d$
$= 2\pi r$
area $= \pi r^2$



Example

Find the circumference and area of a circle of diameter 8 cm. (Take $\pi = 3{\cdot}142$.)

$$\text{Circumference} = \pi d = 3{\cdot}142 \times 8 = 25{\cdot}14\,\text{cm (3 s.f.)}$$

$$\text{Area} = \pi r^2 = 3{\cdot}142 \times 4^2 = 50{\cdot}3\,\text{cm}^2 \text{ (3 s.f.)}$$

Exercise 3

For each shape find (a) the perimeter, (b) the area. All lengths are in cm. Use the π button on a calculator or take $\pi = 3{\cdot}142$. All the arcs are either semi-circles or quarter circles.

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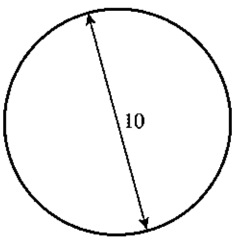

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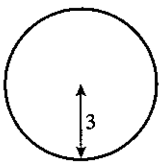

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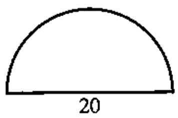

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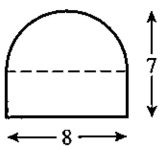

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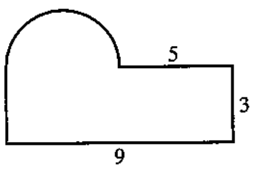

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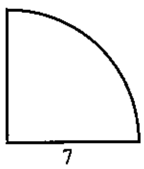

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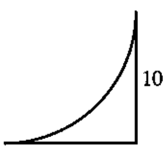

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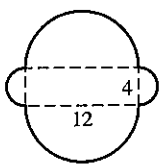

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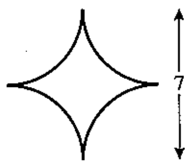

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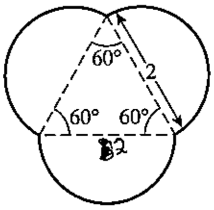

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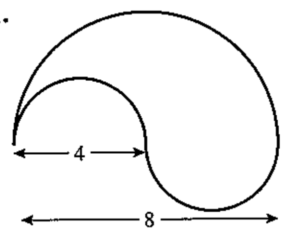

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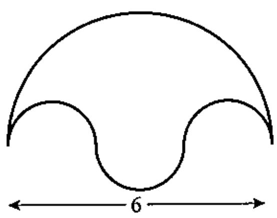


Example 1

A circle has a circumference of 20 m. Find the radius of the circle.

Let the radius of the circle by r m.

$$\text{Circumference} = 2\pi r$$
$$\therefore \quad 2\pi r = 20$$
$$\therefore \quad r = \frac{20}{2\pi}$$
$$r = 3{\cdot}18$$

The radius of the circle is 3·18 m (3 S.F.).

Example 2

A circle has an area of 45 cm^2. Find the radius of the circle.

Let the radius of the circle by r cm.

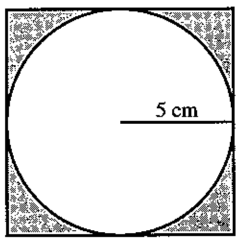
$$\pi r^2 = 45$$
$$r^2 = \frac{45}{\pi}$$
$$r = \sqrt{\left(\frac{45}{\pi}\right)} = 3{\cdot}78 \text{ (3 S.F.)}$$

The radius of the circle is 3·78 cm.

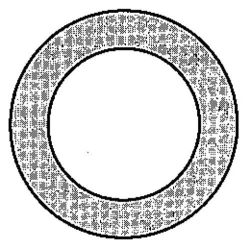
Exercise 4

Use the π button on a calculator and give answers to 3 S.F.

1. A circle has an area of 15 cm^2. Find its radius.
2. A circle has a circumference of 190 m. Find its radius.
3. Find the radius of a circle of area 22 km^2.
4. Find the radius of a circle of circumference 58·6 cm.
5. A circle has an area of 16 mm^2. Find its circumference.
6. A circle has a circumference of 2500 km. Find its area.
7. A circle of radius 5 cm is inscribed inside a square as shown. Find the area shaded.



8. A circular pond of radius 6 m is surrounded by a path of width 1 m.
(a) Find the area of the path.
(b) The path is resurfaced with astroturf which is bought in packs each containing enough to cover an area of 7 m^2. How many packs are required?



9. Discs of radius 4 cm are cut from a rectangular plastic sheet of length 84 cm and width 24 cm.
(a) How many complete discs can be cut out?
Find:
(b) the total area of the discs cut
(c) the area of the sheet wasted.

10. The tyre of a car wheel has an outer diameter of 30 cm. How many times will the wheel rotate on a journey of 5 km?

11. A golf ball of diameter 1·68 inches rolls a distance of 4 m in a straight line. How many times does the ball rotate completely?
(1 inch = 2·54 cm)

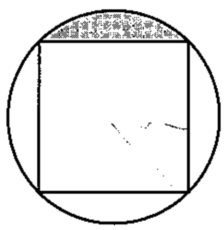
12. 100 yards of cotton is wound without stretching onto a reel of diameter 3 cm. How many times does the reel rotate?
(1 yard = 0·914 m. Ignore the thickness of the cotton.)

13. A rectangular metal plate has a length of 65 cm and a width of 35 cm. It is melted down and recast into circular discs of the same thickness. How many complete discs can be formed if
(a) the radius of each disc is 3 cm?
(b) the radius of each disc is 10 cm?

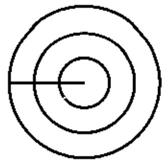
14. Calculate the radius of a circle whose area is equal to the sum of the areas of three circles of radii 2 cm, 3 cm and 4 cm respectively.

15. The diameter of a circle is given as 10 cm, correct to the nearest cm.
Calculate:
(a) the maximum possible circumference
(b) the minimum possible area of the circle consistent with this data.

16. A square is inscribed in a circle of radius 7 cm. Find:
(a) the area of the square
(b) the area shaded.

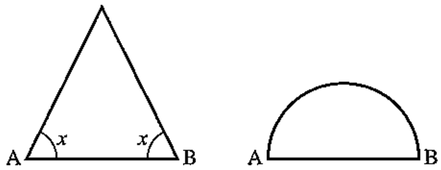


17. An archery target has three concentric regions. The diameters of the regions are in the ratio 1 : 2 : 3. Find the ratio of their areas.



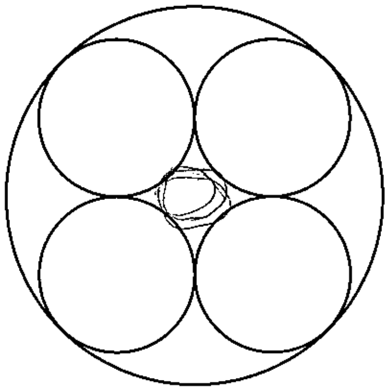
18. The governor of a prison has 100 m of wire fencing. What area can he enclose if he makes a circular compound?

19. The semi-circle and the isosceles triangle have the same base AB and the same area. Find the angle x.

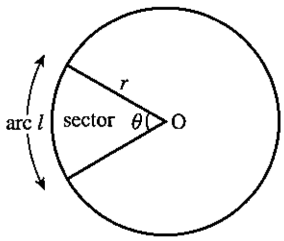


20. Mr Gibson decided to measure the circumference of the Earth using a very long tape measure. For reasons best known to himself he held the tape measure 1 m from the surface of the (perfectly spherical) Earth all the way round. When he had finished Mrs Gibson told him that his measurement gave too large an answer. She suggested taking off 6 m. Was she correct? [Take the radius of the Earth to be 6400 km (if you need it).]

21. The large circle has a radius of 10 cm. Find the radius of the largest circle which will fit in the middle.

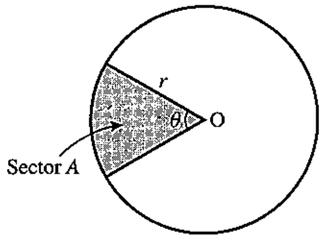


3.3 Arc length and sector area



Arc length, $l = \dfrac{\theta}{360} \times 2\pi r$

We take a fraction of the whole circumference depending on the angle at the centre of the circle.

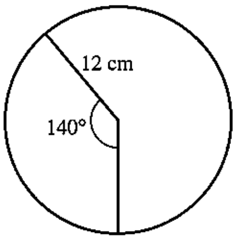


Sector area, $A = \frac{\theta}{360} \times \pi r^2$

We take a fraction of the whole area depending on the angle at the centre of the circle.

Example 1

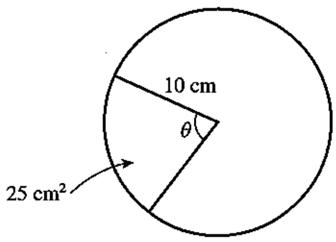
Find the length of an arc which subtends an angle of 140° at the centre of a circle of radius 12 cm. (Take $\pi = \frac{22}{7}$)



$$\text{Arc length} = \frac{140}{360} \times 2 \times \frac{22}{7} \times 12$$
$$= \frac{88}{3}$$
$$= 29\tfrac{1}{3}\,\text{cm}$$

Example 2

A sector of a circle of radius 10 cm has an area of 25 cm^2. Find the angle at the centre of the circle.



Let the angle at the centre of the circle be θ.

$$\frac{\theta}{360} \times \pi \times 10^2 = 25$$

$$\therefore \qquad \theta = \frac{25 \times 360}{\pi \times 100}$$

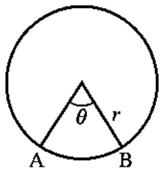
$$\theta = 28{\cdot}6^\circ \text{ (3 s.f.)}$$

The angle at the centre of the circle is 28·6°.

Exercise 5

[Use the π button on a calculator unless told otherwise.]

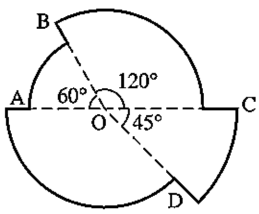
1. Arc AB subtends an angle θ at the centre of circle radius r.



Find the arc length and sector area when:
(a) $r = 4\,\text{cm}$, $\theta = 30^\circ$
(b) $r = 10\,\text{cm}$, $\theta = 45^\circ$
(c) $r = 2\,\text{cm}$, $\theta = 235^\circ$.

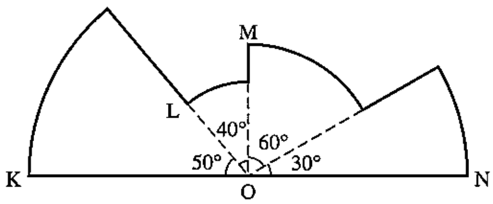
In questions **2** and **3** find the total area of the shape.

2.



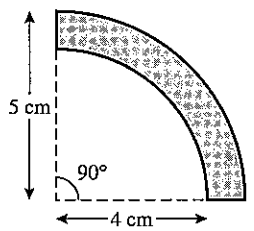
OA = 2 cm, OB = 3 cm, OC = 5 cm, OD = 3 cm.

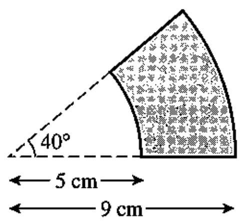
3.



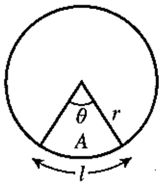
ON = 6 cm, OM = 3 cm, OL = 2 cm, OK = 6 cm.

4. Find the shaded areas.



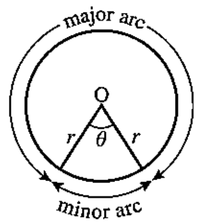


5. In the diagram the arc length is l and the sector area is A.



(a) Find θ, when $r = 5$ cm and $l = 7{\cdot}5$ cm.
(b) Find θ, when $r = 2$ m and $A = 2\,m^2$.
(c) Find r, when $\theta = 55°$ and $l = 6$ cm.

6. The length of the minor arc AB of a circle, centre O, is 2π cm and the length of the major arc is 22π cm. Find:
(a) the radius of the circle
(b) the acute angle AOB.



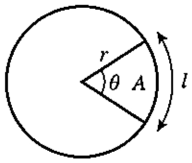
7. The lengths of the minor and major arcs of a circle are 5·2 cm and 19·8 respectively. Find:
(a) the radius of the circle
(b) the angle subtended at the centre by the minor arc.

8. A wheel of radius 10 cm is turning at a rate of 5 revolutions per minute. Calculate:
(a) the angle through which the wheel turns in 1 second
(b) the distance moved by a point on the rim in 2 seconds.

9. The length of an arc of a circle is 12 cm. The corresponding sector area is $108\,cm^2$. Find:
(a) the radius of the circle
(b) the angle subtended at the centre of the circle by the arc.

10. The length of an arc of a circle is 7·5 cm. The corresponding sector area is $37{\cdot}5\,cm^2$. Find:
(a) the radius of the circle
(b) the angle subtended at the centre of the circle by the arc.

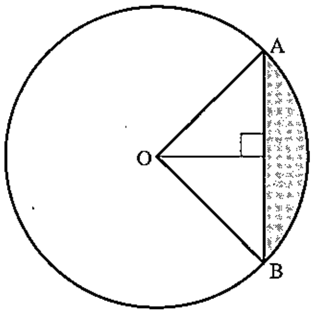
11. In the diagram the arc length is l and the sector area is A.



(a) Find l, when $\theta = 72°$ and $A = 15\,cm^2$.
(b) Find l, when $\theta = 135°$ and $A = 162\,m^2$.
(c) Find A, when $l = 11$ cm and $r = 5{\cdot}2$ cm.

12. A long time ago Mr Gibson found an island shaped as a triangle with three straight shores of length 3 km, 4 km and 5 km. He declared an 'exclusion zone' around his island and forbade anyone to came within 1 km of his shore. What was the area of his exclusion zone?

3.4 Chord of a circle

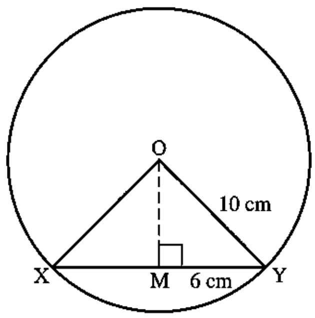


The line AB is a chord. The area of a circle cut off by a chord is called a *segment*. In the diagram the *minor* segment is shaded and the *major* segment is unshaded.

(a) The line from the centre of a circle to the mid-point of a chord *bisects* the chord at *right angles*.
(b) The line from the centre of a circle to the mid-point of a chord bisects the angle subtended by the chord at the centre of the circle.

Example

XY is a chord of length 12 cm of a circle of radius 10 cm, centre O.
Calculate:
(a) the angle XOY
(b) the area of the minor segment cut off by the chord XY.



You can find out more about trigonometry in Unit 4 on page 182.

Let the mid-point of XY be M.

$\therefore \quad MY = 6\,\text{cm}$

$\sin M\hat{O}Y = \dfrac{6}{10}$

$\therefore \quad M\hat{O}Y = 36{\cdot}87°$

$\therefore \quad X\hat{O}Y = 2 \times 36{\cdot}87$
$= 73{\cdot}74°$

area of minor segment = area of sector XOY − area of $\triangle XOY$

area of sector XOY $= \dfrac{73{\cdot}74}{360} \times \pi \times 10^2$
$= 64{\cdot}32\,\text{cm}^2$.

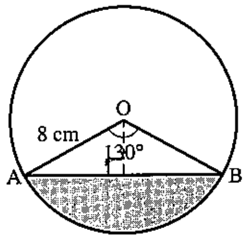
area of $\triangle XOY = \frac{1}{2} \times 10 \times 10 \times \sin 73{\cdot}74°$
$= 48{\cdot}00\,\text{cm}^2$

$\therefore$ Area of minor segment $= 64{\cdot}32 - 48{\cdot}00$
$= 16{\cdot}3\,\text{cm}^2$ (3 S.F.)

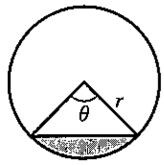
Exercise 6

Use the π button on a calculator. You will need basic trigonometry (page 182).

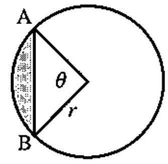
1. The chord AB subtends an angle of 130° at the centre O. The radius of the circle is 8 cm. Find:
   (a) the length of AB
   (b) the area of sector OAB
   (c) the area of triangle OAB
   (d) the area of the minor segment (shown shaded).



2. Find the shaded area when:
   (a) $r = 6\,\text{cm}, \theta = 70°$
   (b) $r = 14\,\text{cm}, \theta = 104°$
   (c) $r = 5\,\text{cm}, \theta = 80°$



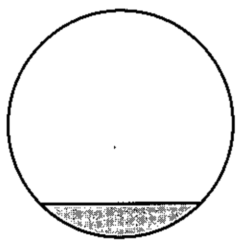
3. Find θ and hence the shaded area when:
   (a) $AB = 10\,\text{cm}, r = 10\,\text{cm}$
   (b) $AB = 8\,\text{cm}, r = 5\,\text{cm}$



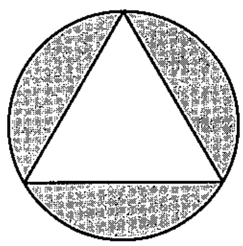
4. How far is a chord of length 8 cm from the centre of a circle of radius 5 cm?

5. How far is a chord of length 9 cm from the centre of a circle of radius 6 cm?

6. The diagram shows the cross-section of a cylindrical pipe with water lying in the bottom.
   (a) If the maximum depth of the water is 2 cm and the radius of the pipe is 7 cm, find the area shaded.
   (b) What is the *volume* of water in a length of 30 cm?



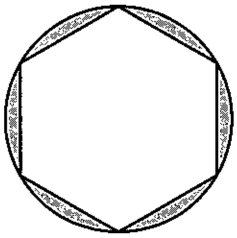
7. An equilateral triangle is inscribed in a circle of radius 10 cm. Find:
   (a) the area of the triangle
   (b) the area shaded.



8. An equilateral triangle is inscribed in a circle of radius 18·8 cm. Find:
   (a) the area of the triangle
   (b) the area of the three segments surrounding the triangle.

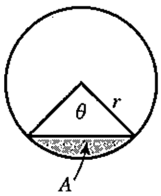
9. A regular hexagon is circumscribed by a circle of radius 6 cm. Find the area shaded.

You can find out more about regular shapes on page 116.

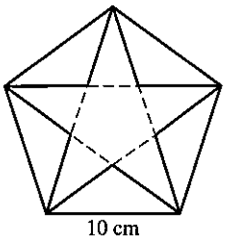


10. A regular octagon is circumscribed by a circle of radius r cm. Find the area enclosed between the circle and the octagon. (Give the answer in terms of r.)

11. Find the radius of the circle:
(a) when $\theta = 90°$, $A = 20\,\text{cm}^2$
(b) when $\theta = 30°$, $A = 35\,\text{cm}^2$
(c) when $\theta = 150°$, $A = 114\,\text{cm}^2$

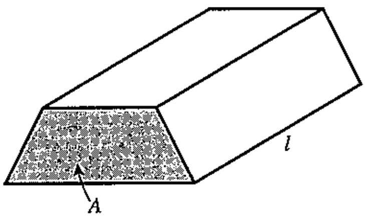


12. (Harder) The diagram shows a regular pentagon of side 10 cm with a star inside. Calculate the area of the star.



3.5 Volume

Prism



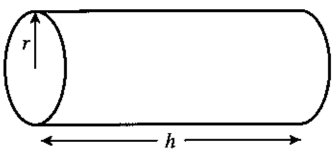
A prism is an object with the same cross-section throughout its length.

Volume of prism = (area of cross-section) × length
$= A \times l$

A *cuboid* is a prism whose six faces are all rectangles. A cube is a special case of a cuboid in which all six faces are squares.

Cylinder

radius $= r$
height $= h$



A cylinder is a prism whose cross-section is a circle.

Volume of cylinder = (area of cross-section) × length
Volume $= \pi r^2 h$

Example

Calculate the height of a cylinder of volume $500\,\text{cm}^3$ and base radius 8 cm. Let the height of the cylinder be h cm.

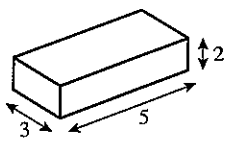
$$\pi r^2 h = 500$$
$$3{\cdot}142 \times 8^2 \times h = 500$$
$$h = \frac{500}{3{\cdot}142 \times 64}$$
$$h = 2{\cdot}49 \text{ (3 s.f.)}$$

The height of the cylinder is 2·49 cm.

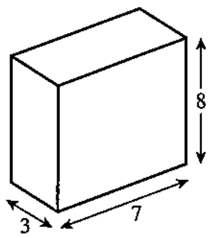
Exercise 7

1. Calculate the volume of the prisms. All lengths are in cm.

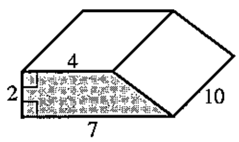
(a)



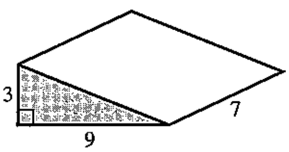
(b)



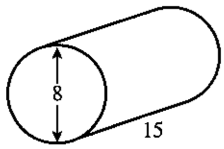
(c)



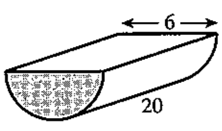
(d)



(e)



(f)



2. Calculate the volume of the following cylinders:
   (a) $r = 4\,\text{cm}$, $h = 10\,\text{cm}$
   (b) $r = 11\,\text{m}$, $h = 2\,\text{m}$
   (c) $r = 2{\cdot}1\,\text{cm}$, $h = 0{\cdot}9\,\text{cm}$

3. Find the height of a cylinder of volume $200\,cm^3$ and radius 4 cm.

4. Find the length of a cylinder of volume 2 litres and radius 10 cm.

5. Find the radius of a cylinder of volume $45\,cm^3$ and length 4 cm.

6. A prism has volume $100\,cm^3$ and length 8 cm. If the cross-section is an equilateral triangle, find the length of a side of the triangle.

7. When 3 litres of oil is removed from an upright cylindrical can the level falls by 10 cm. Find the radius of the can.

8. A solid cylinder of radius 4 cm and length 8 cm is melted down and recast into a solid cube. Find the side of the cube.

9. A solid rectangular block of copper 5 cm by 4 cm by 2 cm is drawn out to make a cylindrical wire of diameter 2 mm. Calculate the length of the wire.

10. Water flows through a circular pipe of internal diameter 3 cm at a speed of 10 cm/s. If the pipe is full, how much water flows from the pipe in one minute? (answer in litres)

11. Water issues from a hose-pipe of internal diameter 1 cm at a rate of 5 litres per minute. At what speed is the water flowing through the pipe?

12. A cylindrical metal pipe has external diameter of 6 cm and internal diameter of 4 cm. Calculate the volume of metal in a pipe of length 1 m. If $1\,cm^3$ of the metal weighs 8 g, find the weight of the pipe.

13. For two cylinders A and B, the ratio of lengths is 3 : 1 and the ratio of diameters is 1 : 2. Calculate the ratio of their volumes.

14. A well-trained hen can lay eggs which are either perfect cylinders of diameter and length 4 cm, or perfect cubes of side 5 cm. Which eggs have the greater volume, and by how much? (Take $\pi = 3$)

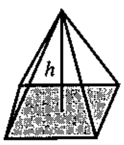
15. Mrs Gibson decided to build a garage and began by calculating the number of bricks required. The garage was to be 6 m by 4 m and 2·5 m in height. Each brick measures 22 cm by 10 cm by 7 cm. Mrs Gibson estimated that she would need about 40 000 bricks. Is this a reasonable estimate?

16. A cylindrical can of internal radius 20 cm stands upright on a flat surface. It contains water to a depth of 20 cm. Calculate the rise in the level of the water when a brick of volume $1500\,cm^3$ is immersed in the water.

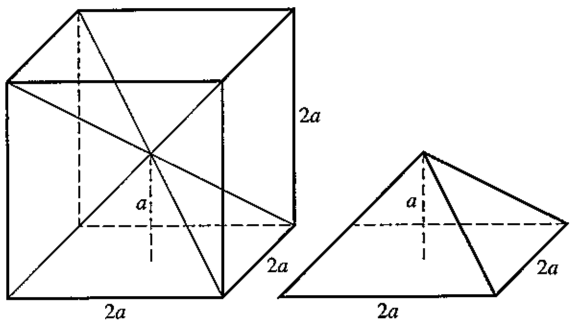
17. A cylindrical tin of height 15 cm and radius 4 cm is filled with sand from a rectangular box. How many times can the tin be filled if the dimensions of the box are 50 cm by 40 cm by 20 cm?

18. Rain which falls onto a flat rectangular surface of length 6 m and width 4 m is collected in a cylinder of internal radius 20 cm. What is the depth of water in the cylinder after a storm in which 1 cm of rain fell?

Pyramid



Volume $= \frac{1}{3}$(base area) × height.



(i) (ii)

Figure (i) shows a cube of side $2a$ broken down into six pyramids of height a as shown in figure (ii).

If the volume of each pyramid is V,

then $6V = 2a \times 2a \times 2a$

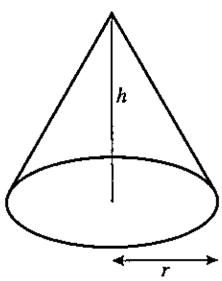
$V = \frac{1}{6} \times (2a)^2 \times 2a$

so $V = \frac{1}{3} \times (2a)^2 \times a$

$V = \frac{1}{3}$(base area) × height

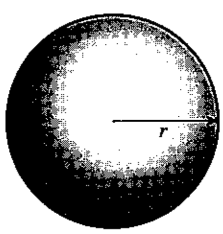
Cone

Volume $= \frac{1}{3}\pi r^2 h$
(note the similarity with the pyramid)



Sphere

Volume $= \frac{4}{3}\pi r^3$



Example 1

A pyramid has a square base of side 5 m and vertical height 4 m. Find its volume.

Volume of pyramid $= \frac{1}{3}(5 \times 5) \times 4$
$= 33\frac{1}{3}\ \text{m}^3$

Example 2

Calculate the radius of a sphere of volume $500\ \text{cm}^3$.

Let the radius of the sphere be r cm

$\frac{4}{3}\pi r^3 = 500$

$r^3 = \dfrac{3 \times 500}{4\pi}$

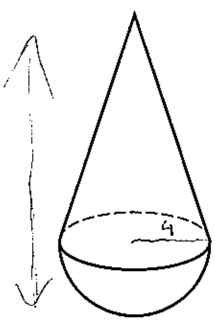
$r = \sqrt[3]{\left(\dfrac{3 \times 500}{4\pi}\right)} = 4{\cdot}92$ (3 S.F.)

The radius of the sphere is 4·92 cm.

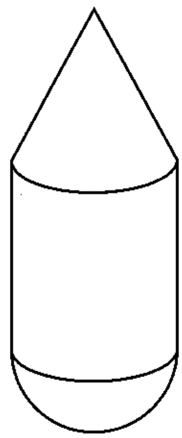
Exercise 8

Find the volumes of the following objects:

1. cone: height = 5 cm, radius = 2 cm
2. sphere: radius = 5 cm
3. sphere: radius = 10 cm
4. cone: height = 6 cm, radius = 4 cm
5. sphere: diameter = 8 cm
6. cone: height = x cm, radius = $2x$ cm
7. sphere: radius = 0·1 m
8. cone: height = $\dfrac{1}{\pi}$ cm, radius = 3 cm
9. pyramid: rectangular base 7 cm by 8 cm; height = 5 cm
10. pyramid: square base of side 4 m, height = 9 m
11. pyramid: equilateral triangular base of side = 8 cm, height = 10 cm
12. Find the volume of a hemisphere of radius 5 cm.
13. A cone is attached to a hemisphere of radius 4 cm. If the total height of the object is 10 cm, find its volume.



14. A toy consists of a cylinder of diameter 6 cm 'sandwiched' between a hemisphere and a cone of the same diameter. If the cone is of height 8 cm and the cylinder is of height 10 cm, find the total volume of the toy.



15. Find the height of a pyramid of volume $20\,m^3$ and base area $12\,m^2$.

16. Find the radius of a sphere of volume $60\,cm^3$.

17. Find the height of a cone of volume 2·5 litre and radius 10 cm.

18. Six square-based pyramids fit exactly onto the six faces of a cube of side 4 cm. If the volume of the object formed is $256\,cm^3$, find the height of each of the pyramids.

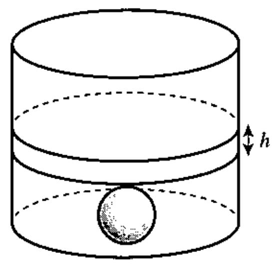
19. A solid metal cube of side 6 cm is recast into a solid sphere. Find the radius of the sphere.

20. A hollow spherical vessel has internal and external radii of 6 cm and 6·4 cm respectively. Calculate the weight of the vessel if it is made of metal of density $10\,g/cm^3$.

21. Water is flowing into an inverted cone, of diameter and height 30 cm, at a rate of 4 litres per minute. How long, in seconds, will it take to fill the cone?

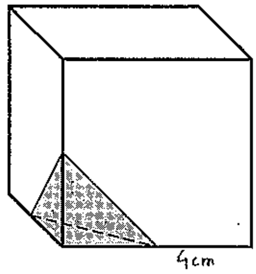
22. A solid metal sphere is recast into many smaller spheres. Calculate the number of the smaller spheres if the initial and final radii are as follows:
(a) initial radius = 10 cm, final radius = 2 cm
(b) initial radius = 7 cm, final radius = $\frac{1}{2}$ cm
(c) initial radius = 1 m, final radius = $\frac{1}{3}$ cm.

23. A spherical ball is immersed in water contained in a vertical cylinder.
Assuming the water covers the ball, calculate the rise in the water level if:
(a) sphere radius = 3 cm, cylinder radius = 10 cm
(b) sphere radius = 2 cm, cylinder radius = 5 cm.



24. A spherical ball is immersed in water contained in a vertical cylinder. The rise in water level is measured in order to calculate the radius of the spherical ball. Calculate the radius of the ball in the following cases:
(a) cylinder of radius 10 cm, water level rises 4 cm.
(b) cylinder of radius 100 cm, water level rises 8 cm.

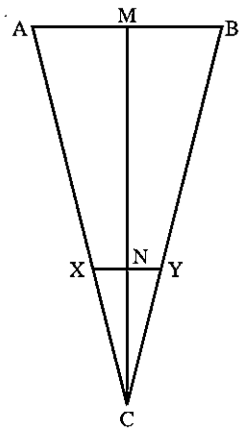
25. One corner of a solid cube of side 8 cm is removed by cutting through the mid-points of three adjacent sides. Calculate the volume of the piece removed.



26. The cylindrical end of a pencil is sharpened to produce a perfect cone at the end with no overall loss of length. If the diameter of the pencil is 1 cm, and the cone is of length 2 cm, calculate the volume of the shavings.

27. Metal spheres of radius 2 cm are packed into a rectangular box of internal dimensions 16 cm × 8 cm × 8 cm. When 16 spheres are packed the box is filled with a preservative liquid. Find the volume of this liquid.

28. The diagram shows the cross-section of an inverted cone of height MC = 12 cm. If AB = 6 cm and XY = 2 cm, use similar triangles to find the length NC.
(You can find out about similar triangles on page 116.)

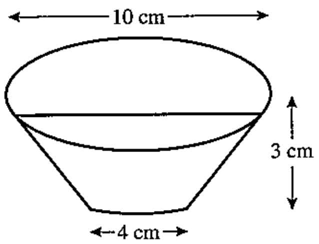


29. An inverted cone of height 10 cm and base radius 6·4 cm contains water to a depth of 5 cm, measured from the vertex. Calculate the volume of water in the cone.

30. An inverted cone of height 15 cm and base radius 4 cm contains water to a depth of 10 cm. Calculate the volume of water in the cone.

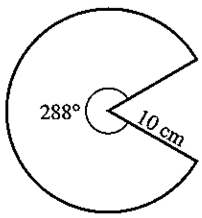
31. An inverted cone of height 12 cm and base radius 6 cm contains 20 cm^3 of water. Calculate the depth of water in the cone, measured from the vertex.

32. A frustum is a cone with 'the end chopped off'. A bucket in the shape of a frustum as shown has diameters of 10 cm and 4 cm at its ends and a depth of 3 cm. Calculate the volume of the bucket.

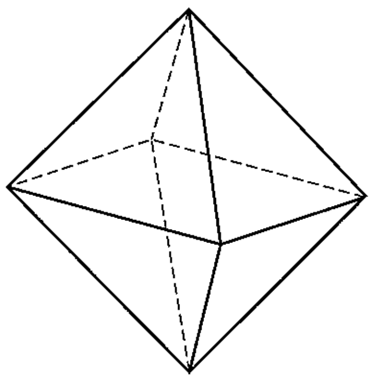


33. Find the volume of a frustum with end diameters of 60 cm and 20 cm and a depth of 40 cm.

34. The diagram shows a sector of a circle of radius 10 cm.
(a) Find, as a multiple of π, the arc length of the sector.
The straight edges are brought together to make a cone. Calculate:
(b) the radius of the base of the cone,
(c) the vertical height of the cone.



35. Calculate the volume of a regular octahedron whose edges are all 10 cm.



36. A sphere passes through the eight corners of a cube of side 10 cm. Find the volume of the sphere.

37. (Harder) Find the volume of a regular tetrahedron of side 20 cm.

38. Find the volume of a regular tetrahedron of side 35 cm.

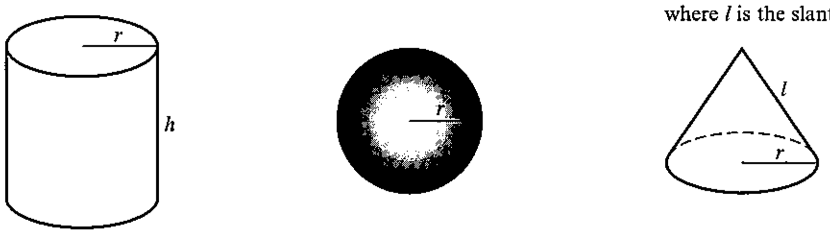
3.6 Surface area

We are concerned here with the surface areas of the *curved* parts of cylinders, spheres and cones. The areas of the plane faces are easier to find.

(a) Cylinder
Curved surface area $= 2\pi rh$

(b) Sphere
Surface area $= 4\pi r^2$

(c) Cone
Curved surface area $= \pi rl$
where l is the slant height.

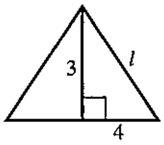


Example

Find the *total* surface area of a solid cone of radius 4 cm and vertical height 3 cm.

Let the slant height of the cone be l cm.

$l^2 = 3^2 + 4^2$ (Pythagoras' Theorem)
$l = 5$



$$\begin{aligned}\text{Curved surface area} &= \pi \times 4 \times 5 \\ &= 20\pi\,\text{cm}^2 \\ \text{Area of end face} &= \pi \times 4^2 = 16\pi\,\text{cm}^2 \\ \therefore \quad \text{Total surface area} &= 20\pi + 16\pi \\ &= 36\pi\,\text{cm}^2 \\ &= 113\,\text{cm}^2 \text{ to 3 s.f.}\end{aligned}$$

Exercise 9

Use the π button on a calculator unless otherwise instructed.

1. Copy the table and find the quantities marked *.
(Leave π in your answers.)

	solid object	radius	vertical height	curved surface area	total surface area
(a)	sphere	3 cm		*	
(b)	cylinder	4 cm	5 cm		*
(c)	cone	6 cm	8 cm	*	
(d)	cylinder	0·7 m	1 m		*
(e)	sphere	10 m		*	
(f)	cone	5 cm	12 cm	*	
(g)	cylinder	6 mm	10 mm		*
(h)	cone	2·1 cm	4·4 cm	*	
(i)	sphere	0·01 m		*	
(j)	hemisphere	7 cm		*	*

2. Find the radius of a sphere of surface area $34\,cm^2$.

3. Find the slant height of a cone of curved surface area $20\,cm^2$ and radius 3 cm.

4. Find the height of a solid cylinder of radius 1 cm and *total* surface area $28\,cm^2$.

5. Copy the table and find the quantities marked *. (Take $\pi = 3$)

object	radius	vertical height	curved surface area	total surface area
(a) cylinder	4 cm	*	$72\,cm^2$	
(b) sphere	*		$192\,cm^2$	
(c) cone	4 cm	*	$60\,cm^2$	
(d) sphere	*		$0{\cdot}48\,m^2$	
(e) cylinder	5 cm	*		$330\,cm^2$
(f) cone	6 cm	*		$225\,cm^2$
(g) cylinder	2 m	*		$108\,m^2$

6. A solid wooden cylinder of height 8 cm and radius 3 cm is cut in two along a vertical axis of symmetry. Calculate the total surface area of the two pieces.

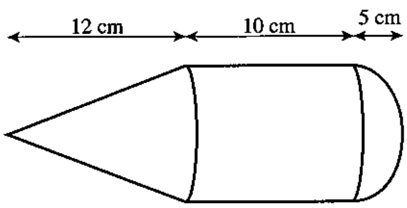
7. A tin of paint covers a surface area of $60\,m^2$ and costs \$4·50. Find the cost of painting the outside surface of a hemispherical dome of radius 50 m. (Just the curved part.)

8. A solid cylinder of height 10 cm and radius 4 cm is to be plated with material costing \$11 per cm^2. Find the cost of the plating.

9. Find the volume of a sphere of surface area $100\,cm^2$.

10. Find the surface area of a sphere of volume $28\,cm^3$.

11. Calculate the total surface area of the combined cone/cylinder/hemisphere.

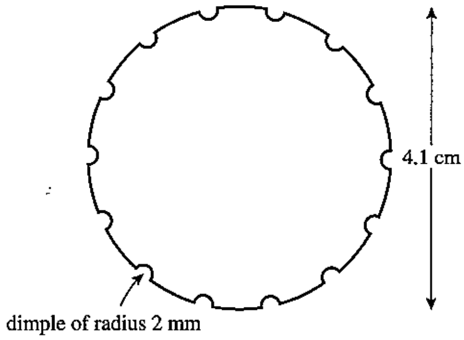


12. A man is determined to spray the entire surface of the Earth (including the oceans) with a revolutionary new weed killer. If it takes him 10 seconds to spray $1\,m^2$, how long will it take to spray the whole world? (radius of the Earth = 6370 km; ignore leap years)

13. An inverted cone of vertical height 12 cm and base radius 9 cm contains water to a depth of 4 cm. Find the area of the interior surface of the cone not in contact with the water.

14. A circular paper of radius 20 cm is cut in half and each half is made into a hollow cone by joining the straight edges. Find the slant height and base radius of each cone.

15. A golf ball has a diameter of 4·1 cm and the surface has 150 dimples of radius 2 mm. Calculate the total surface area which is exposed to the surroundings. (Assume the 'dimples' are hemispherical.)

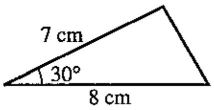


16. A cone of radius 3 cm and slant height 6 cm is cut into four identical pieces. Calculate the total surface area of the four pieces.

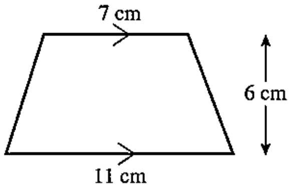
Revision exercise 3A

1. Find the area of the following shapes:

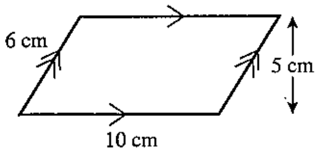
(a)



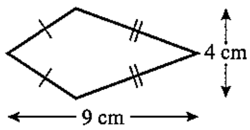
(b)



(c)

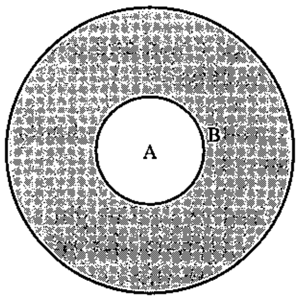


(d)



2. (a) A circle has radius 9 m. Find its circumference and area.
(b) A circle has circumference 34 cm. Find its diameter.
(c) A circle has area 50 cm^2. Find its radius.

3. A target consists of concentric circles of radii 3 cm and 9 cm.
(a) Find the area of A, in terms of π.
(b) Find the ratio $\dfrac{\text{area of B}}{\text{area of A}}$.



4. In Figure 1 a circle of radius 4 cm is inscribed in a square. In Figure 2 a square is inscribed in a circle of radius 4 cm.
Calculate the shaded area in each diagram.

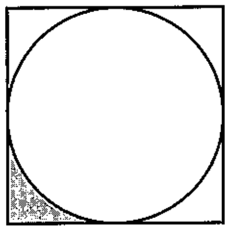


Figure 1

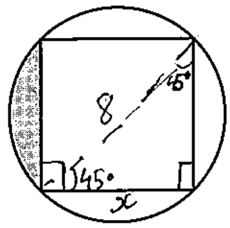
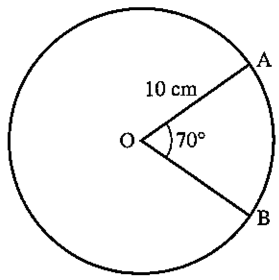


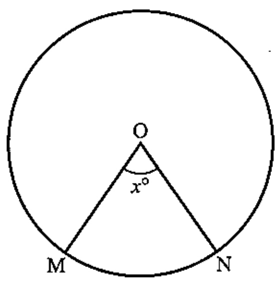
Figure 2

5. Given that OA = 10 cm and $A\hat{O}B = 70°$ (where O is the centre of the circle), calculate:
(a) the arc length AB
(b) the area of minor sector AOB.



6. The points X and Y lie on the circumference of a circle, of centre O and radius 8 cm, where $X\hat{O}Y = 80°$. Calculate:
(a) the length of the minor arc XY
(b) the length of the chord XY
(c) the area of sector XOY
(d) the area of triangle XOY
(e) the area of the minor segment of the circle cut off by XY.

7. Given that ON = 10 cm and minor arc MN = 18 cm, calculate the angle $M\hat{O}N$ (shown as $x°$).



8. A cylinder of radius 8 cm has a volume of 2 litres. Calculate the height of the cylinder.

9. Calculate:
(a) the volume of a sphere of radius 6 cm
(b) the radius of a sphere whose volume is 800 cm^3.

10. A sphere of radius 5 cm is melted down and made into a solid cube. Find the length of a side of the cube.

11. The curved surface area of a solid circular cylinder of height 8 cm is 100 cm^2. Calculate the volume of the cylinder.

12. A cone has base radius 5 cm and vertical height 10 cm, correct to the nearest cm. Calculate the maximum and minimum possible volumes of the cone, consistent with this data.

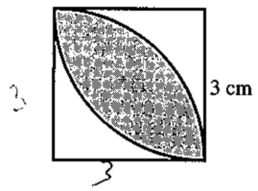
13. Calculate the radius of a hemispherical solid whose total surface area is $48\pi\,cm^2$.

14. Calculate:
(a) the area of an equilateral triangle of side 6 cm.
(b) the area of a regular hexagon of side 6 cm.
(c) the volume of a regular hexagonal prism of length 10 cm, where the side of the hexagon is 12 cm.

15. Ten spheres of radius 1 cm are immersed in liquid contained in a vertical cylinder of radius 6 cm. Calculate the rise in the level of the liquid in the cylinder.

16. A cube of side 10 cm is melted down and made into ten identical spheres. Calculate the surface area of one of the spheres.

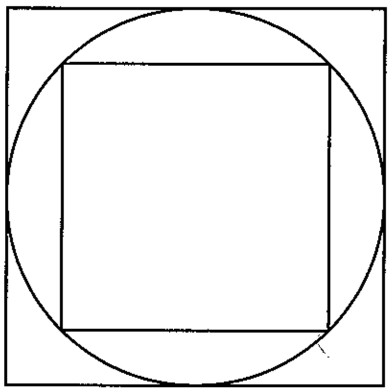
17. The square has sides of length 3 cm and the arcs have centres at the corners. Find the shaded area.



18. A copper pipe has external diameter 18 mm and thickness 2 mm. The density of copper is $9\,g/cm^3$ and the price of copper is \$150 per tonne. What is the cost of the copper in a length of 5 m of this pipe?

19. Twenty-seven small wooden cubes fit exactly inside a cubical box without a lid. How many of the cubes are touching the sides or the bottom of the box?

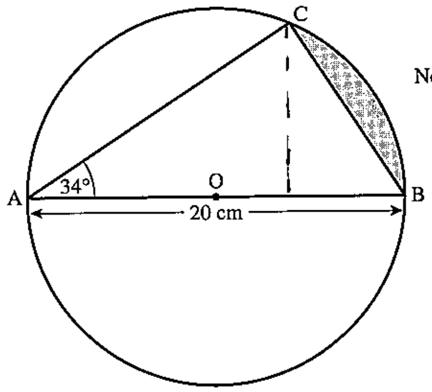
20. In the diagram the area of the smaller square is $10\,cm^2$. Find the area of the larger square.



Examination exercise 3B

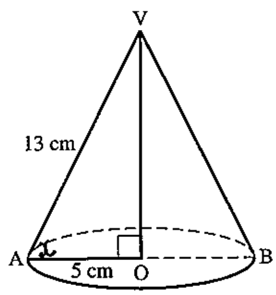
You will need basic trigonometry for this exercise (page 182).

1.

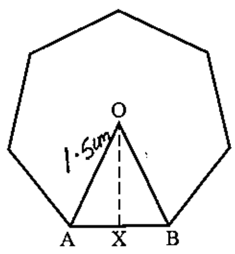


A circular plate, centre O, has diameter AOB = 20 cm.
C is a point on the circumference such that angle CAB = 34°.
A crack appears along the chord CB and the shaded segment drops off. Calculate:
(a) the length of the chord CB,
(b) the acute angle COB,
(c) the length of the minor arc CB. [π is approximately 3·142.]
J 97 2

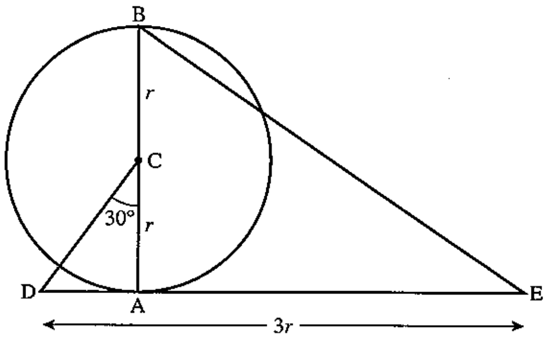
2. The diagram shows a cone with base diameter AOB and vertical height OV. AO = 5 cm and slant height AV = 13 cm. Calculate:
(a) angle VAO,
(b) the curved surface area of the cone.
[The curved surface area of a cone, base radius r and slant height l, is πrl.]
[π is approximately 3·142.]
J 96 2



3. The diagram shows the face of a coin. It is a regular seven-sided polygon. O is the centre of the face of the coin and AO = 1·5 centimetres.
(a) Show that, correct to two decimal places, angle OAB = 64·29°.
(b) Giving your answers correct to three significant figures, calculate:
(i) the length of the perpendicular, OX, from O to AB,
(ii) the length of AB,
(iii) the area of triangle AOB,
(iv) the area of the whole face.
(c) The coin is 3 millimetres thick. Calculate its volume.
N 97 4



4.



BCA is the diameter of a circle, centre C, radius r. DAE is a tangent to the circle at A. DE $= 3r$ and $D\hat{C}A = 30°$.

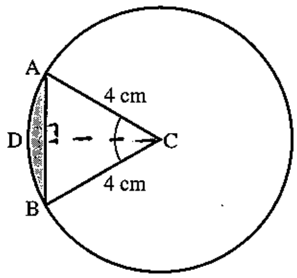
(a) (i) Draw the diagram accurately when $r = 3$ cm.
  (ii) Measure and write down the length of BE in your diagram.
  (iii) Calculate the length of the semicircular arc BA when $r = 3$ cm. [π is approximately 3·142.]

(b) In the case when $r = 10$ cm, calculate, to two decimal places:
  (i) the length of DA,
  (ii) the length of AE,
  (iii) the length of BE,
  (iv) the length of the semicircular arc BA.

(c) Comment on the relationship between the length of BE and the length of the semicircular arc BA. J 95 4

5. (a) A circle, centre C, has radius 4 cm.

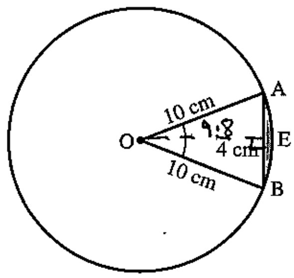
The length of the arc ADB is $\frac{4\pi}{3}$ cm.



Not to scale

  (i) Show that angle ACB $= 60°$.
  (ii) Calculate the area of sector ACBD. [π is approximately 3·142.]
  (iii) Calculate the area of triangle ACB.
  (iv) Find the area of the shaded segment ADB, correct to two decimal places.

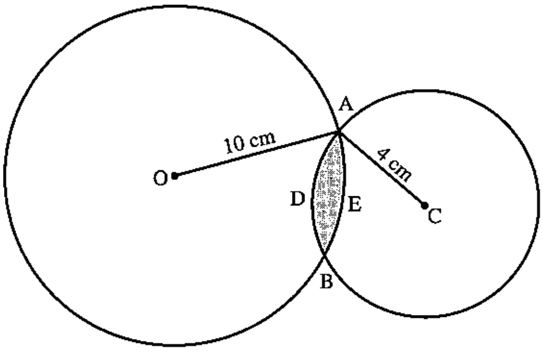
(b) A circle, centre O, has radius 10 cm.
The chord AB $= 4$ cm.



Not to scale

  (i) Show that angle AOB $= 23·1°$, to the nearest 0·1°.
  (ii) Calculate the area of the shaded segment AEB.

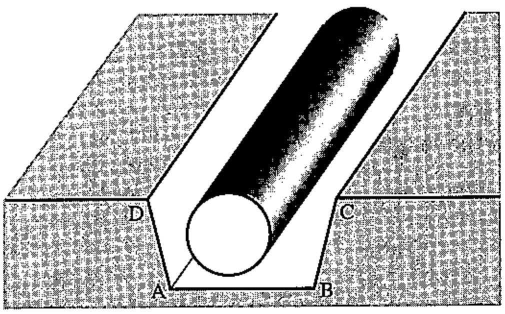
(c)



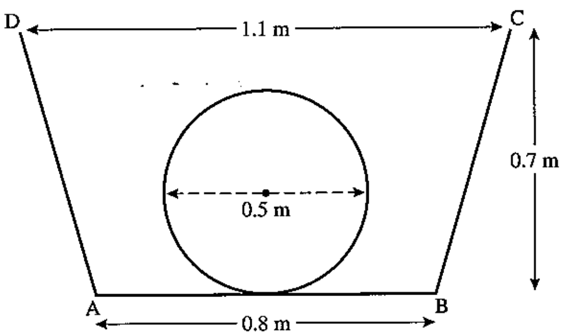
Not to scale

The two circles in parts (a) and (b) are placed together so that they intersect at A and B. Write down the shaded area enclosed by the arcs ADB and AEB. J 96 4

6.



The diagram shows a trench which has been dug out of level ground so that a cylindrical water pipe can be laid.



Not to scale

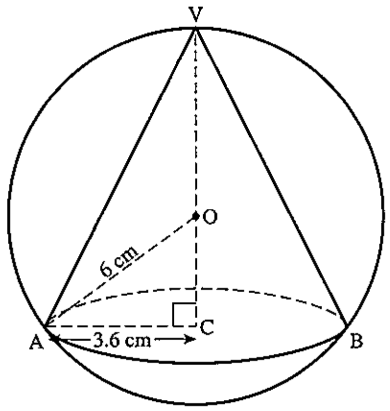
The cross-section, ABCD, of the trench is a trapezium with horizontal sides of length 1·1 m and 0·8 m and height 0·7 m.
The length of the trench is 500 m.

(a) Calculate the volume of earth removed.
(b) If 1 m^3 of earth has a mass of 1·8 tonnes, calculate the mass of earth removed.

(c) The diameter of the pipe is 0·5 m.
After the pipe has been laid earth is replaced until the ground is again level. Calculate the percentage of the earth which is not replaced. [π is approximately 3·142.]

(d) If water flows through the pipe at 0·8 m/s, how many litres will flow through the pipe in 1 hour?
[$1\,m^3 = 1000$ litres] N 95 4

7.



A child's toy consists of a cone inside a sphere. The radius of the sphere, OA, is 6 cm and the radius of the base of the cone, AC, is 3·6 cm.
[The volume of a sphere of radius R is $\frac{4}{3}\pi R^3$.]
[The volume of a cone of base radius r and height h is $\frac{1}{3}\pi r^2 h$.]
[π is approximately 3·142.]

(a) Show that VOC, the height of the cone, is 10·8 cm.

(b) Calculate:
 (i) the volume of the sphere,
 (ii) the volume of the cone,
 (iii) the percentage of the volume of the sphere not occupied by the cone.

(c) The sphere rolls 3 m across the floor in a straight line.
Calculate:
 (i) the circumference of the sphere,
 (ii) the number of complete revolutions made by the sphere,
 (iii) the number of degrees through which the sphere must still turn in order to complete another revolution. J 97 4