2 ALGEBRA 1

Isaac Newton (1642–1727) is thought by many to have been one of the greatest intellects of all time. He went to Trinity College Cambridge in 1661 and by the age of 23 he had made three major discoveries: the nature of colours, the calculus and the law of gravitation. He used his version of the calculus to give the first satisfactory explanation of the motion of the Sun, the Moon and the stars. Because he was extremely sensitive to criticism, Newton was always very secretive, but he was eventually persuaded to publish his discoveries in 1687.

3 Use directed numbers in practical situations

20 Substitute numbers in formulae; construct and transform more complicated formulae and equations

21 Manipulate directed numbers; expand products of algebraic expressions; factorise expressions

24 Solve simple linear equations in one unknown; solve simultaneous linear equations in two unknowns; solve quadratic equations by factorisation and use of the formula

2.1 Directed numbers

To add two directed numbers with the same sign, find the sum of the numbers and give the answer the same sign.

Example 1

$+3 + (+5) = +3 + 5 = +8$
$-7 + (-3) = -7 - 3 = -10$
$-9{\cdot}1 + (-3{\cdot}1) = 9{\cdot}1 - 3{\cdot}1 = -12{\cdot}2$
$-2 + (-1) + (-5) = (-2 - 1) - 5$
$= -3 - 5$
$= -8$

To add two directed numbers with different signs, find the difference between the numbers and give the answer the sign of the larger number.

Example 2

$+7 + (-3) = +7 - 3 = +4$
$+9 + (-12) = +9 - 12 = 3$
$-8 + (+4) = -8 + 4 = -4$

To subtract a directed number, change its sign and add.

Example 3

$+7 - (+5) = +7 - 5 = +2$
$+7 - (-5) = +7 + 5 = +12$
$-8 - (+4) = -8 - 4 = -12$
$-9 - (-11) = -9 + 11 = +2$

Exercise 1

1. $+7 + (+6)$
2. $+11 + (+200)$
3. $-3 + (-9)$
4. $-7 + (-24)$
5. $-5 + (-61)$
6 $+0{\cdot}2 + (+5{\cdot}9)$
7. $+5 + (+4{\cdot}1)$
8. $-8 + (-27)$
9. $+17 + (+1{\cdot}7)$
10. $-2 + (-3) + (-4)$
11. $-7 + (+4)$
12. $+7 + (-4)$
13. $-9 + (+7)$
14. $+16 + (-30)$
15. $+14 + (-21)$
16. $-7 + (+10)$
17. $-19 + (+200)$
18. $+7{\cdot}6 + (-9{\cdot}8)$
19. $-1{\cdot}8 + (+10)$
20. $-7 + (+24)$
21. $+7 - (+5)$
22. $+9 - (+15)$
23. $-6 - (+9)$
24. $-9 - (+5)$
25. $+8 - (+10)$
26. $-19 - (-7)$
27. $-10 - (+70)$
28. $-5{\cdot}1 - (+8)$
29. $-0{\cdot}2 - (+4)$
30. $+5{\cdot}2 - (-7{\cdot}2)$
31. $-4 + (-3)$
32. $+6 - (-2)$
33. $+8 + (-4)$
34. $-4 - (+6)$
35. $+7 - (-4)$
36. $+6 + (-2)$
37. $+10 - (+30)$
38. $+19 - (+11)$
39. $+4 + (-7) + (-2)$
40. $-3 - (+2) + (-5)$
41. $-17 - (-1) + (-10)$
42. $-5 + (-7) - (+9)$
43. $+9 + (-7) - (-6)$
44. $-7 - (-8)$
45. $-10{\cdot}1 + (-10{\cdot}1)$
46. $-75 - (-25)$
47. $-204 - (+304)$
48. $-7 + (-11) - (+11)$
49. $+17 - (+17)$
50. $-6 + (-7) - (+8)$
51. $+7 + (-7{\cdot}1)$
52. $-11 - (-4) + (+3)$
53. $-2 - (-8{\cdot}7)$
54. $+7 + (-11) + (+5)$
55. $-610 + (-240)$
56. $-7 - (-3) - (-8)$
57. $+9 - (-6) + (-9)$
58. $-1 - (-5) + (-8)$
59. $-2{\cdot}1 + (-9{\cdot}9)$
60. $-47 - (-16)$

When two directed numbers with the same sign are multiplied together, the answer is positive.

- $+7 \times (+3) = +21$
- $-6 \times (-4) = +24$

When two directed numbers with different signs are multiplied together, the answer is negative.

- $-8 \times (+4) = -32$
- $+7 \times (-5) = -35$
- $-3 \times (+2) \times (+5) = -6 \times (+5) = -30$

When dividing directed numbers, the rules are the same as in multiplication.

- $-70 \div (-2) = +35$
- $+12 \div (-3) = -4$
- $-20 \div (+4) = -5$

Exercise 2

1. $+2 \times (-4)$
2. $+7 \times (+4)$
3. $-4 \times (-3)$
4. $-6 \times (-4)$
5. $-6 \times (-3)$
6. $+5 \times (-7)$
7. $-7 \times (-7)$
8. $-4 \times (+3)$
9. $+0{\cdot}5 \times (-4)$
10. $-1\frac{1}{2} \times (-6)$
11. $-8 \div (+2)$
12. $+12 \div (+3)$
13. $+36 \div (-9)$
14. $-40 \div (-5)$
15. $-70 \div (-1)$
16. $-56 \div (+8)$
17. $-\frac{1}{2} \div (-2)$
18. $-3 \div (+5)$
19. $+0{\cdot}1 \div (-10)$
20. $-0{\cdot}02 \div (-100)$
21. $-11 \times (-11)$
22. $-6 \times (-1)$
23. $+12 \times (-50)$
24. $-\frac{1}{2} \div (+\frac{1}{2})$
25. $-600 \div (+30)$
26. $-5{\cdot}2 \div (+2)$
27. $+7 \times (-100)$
28. $-6 \div (-\frac{1}{3})$
29. $100 \div (-0{\cdot}1)$
30. -8×-80
31. $-3 \times (-2) \times (-1)$
32. $+3 \times (-7) \times (+2)$
33. $+0{\cdot}4 \div (-1)$
34. $-16 \div (+40)$
35. $+0{\cdot}2 \times (-1000)$
36. $-7 \times (-5) \times (-1)$
37. $-14 \div (+7)$
38. $-7 \div (-14)$
39. $+1\frac{1}{4} \div (-5)$
40. $-6 \times (-\frac{1}{2}) \times (-30)$

Exercise 3

1. $-7 + (-3)$
2. $-6 - (-7)$
3. $-4 \times (-3)$
4. $-4 \times (+7)$
5. $4 - (+6)$
6. $-4 \times (-4)$
7. $+6 \div (-2)$
8. $+8 - (-6)$
9. $-7 \times (+4)$
10. $-8 \div (-2)$
11. $+10 \div (-60)$
12. (-3^2)
13. $40 - (+70)$
14. $-6 \times (-4)$
15. $(-1)^5$
16. $-8 \div (+4)$
17. $+10 \times (-3)$
18. $-7 \times (-1)$
19. $+10 + (-7)$
20. $+12 - (-4)$
21. $+100 + (-7)$
22. $-60 \times (-40)$
23. $-20 \div (-2)$
24. $(-1)^{20}$
25. $6 - (+10)$
26. $-6 \times (+4) \times (-2)$
27. $+8 \div (-8)$
28. $0 \times (-6)$
29. $(-2)^3$
30. $+100 - (-70)$
31. $+18 \div (-6)$
32. $(-1)^{12}$
33. $-6 - (-7)$
34. $(-2)^2 + (-4)$
35. $+8 - (-7)$
36. $+7 + (-2)$
37. $-6 \times (+0{\cdot}4)$
38. $-3 \times (-6) \times (-10)$
39. $(-2)^2 + (+1)$
40. $+6 - (+1000)$
41. $(-3)^2 - 7$
42. $-12 \div \frac{1}{4}$
43. $-30 \div -\frac{1}{2}$
44. $5 - (+7) + (-0{\cdot}5)$
45. $(-2)^5$
46. $0 \div (-\frac{1}{5})$
47. $(-0{\cdot}1)^2 \times (-10)$
48. $3 - (+19)$
49. $2{\cdot}1 + (-6{\cdot}4)$
50. $(-\frac{1}{2})^2 \div (-4)$

2.2 Formulae

When a calculation is repeated many times it is often helpful to use a formula. Publishers use a formula to work out the selling price of a book based on the production costs and the expected sales of the book.

Exercise 4

1. The final speed v of a car is given by the formula $v = u + at$.
[u = initial speed, a = acceleration, t = time taken]
Find v when $u = 15\,\text{m/s}$, $a = 0{\cdot}2\,\text{m/s}^2$, $t = 30\,\text{s}$.

2. The time period T of a simple pendulum is given by the formula

$T = 2\pi\sqrt{\left(\frac{l}{g}\right)}$, where l is the length of the pendulum and g is the gravitational acceleration. Find T when $l = 0.65\,\text{m}$, $g = 9.81\,\text{m/s}^2$ and $\pi = 3{\cdot}142$.

3. The total surface area A of a cone is related to the radius r and the slant height l by the formula $A = \pi r(r + l)$. Find A when $r = 7$ cm and $l = 11$ cm.

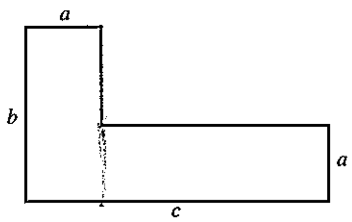
4. The sum S of the squares of the integers from 1 to n is given by $S = \frac{1}{6}n(n + 1)(2n + 1)$. Find S when $n = 12$.

5. The acceleration a of a train is found using the formula $a = \dfrac{v^2 - u^2}{2s}$.
Find a when $v = 20$ m/s, $u = 9$ m/s and $s = 2{\cdot}5$ m.

6. Einstein's famous equation relating energy, mass and the speed of light is $E = mc^2$. Find E when $m = 0{\cdot}0001$ and $c = 3 \times 10^8$.

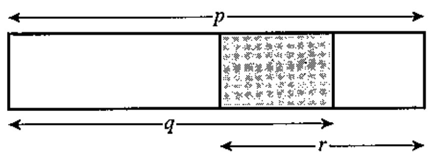
7. The distance s travelled by an accelerating rocket is given by $s = ut + \frac{1}{2}at^2$. Find s when $u = 3$ m/s, $t = 100$ s and $a = 0{\cdot}1$ m/s^2.

8. Find a formula for the area of the shape opposite, in terms of a, b and c.

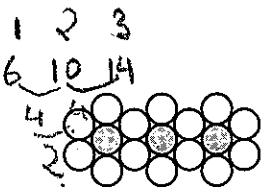


You can find out more about area in Unit 3 on page 82.

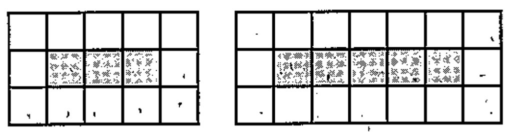
9. Find a formula for the length of the shaded part below, in terms of p, q and r.



10. An intelligent fish lays brown eggs or white eggs and it likes to lay them in a certain pattern. Each brown egg is surrounded by six white eggs.
Here there are 3 brown eggs and 14 white eggs.
(a) How many eggs does it lay altogether if it lays 200 brown eggs?
(b) How many eggs does it lay altogether if it lays n brown eggs?



11. In the diagrams below the rows of black tiles are surrounded by white tiles.



Find a formula for the number of white tiles which would be needed to surround a row of n black tiles.

Example

When $a = 3$, $b = -2$, $c = 5$, find the value of:

(a) $3a + b$ (b) $ac + b^2$ (c) $\frac{a+c}{b}$ (d) $a(c - b)$

(a) $3a + b$
$= (3 \times 3) + (-2)$
$= 9 - 2$
$= 7$

(b) $ac + b^2$
$= (3 \times 5) + (-2)^2$
$= 15 + 4$
$= 19$

(c) $\frac{a+c}{b}$
$= \frac{3+5}{-2}$
$= \frac{8}{-2}$
$= -4$

(d) $a(c - b)$
$= 3[5 - (-2)]$
$= 3[7]$
$= 21$

Notice that working *down* the page is often easier to follow.

Exercise 5

Evaluate the following:
For questions **1** to **12**, $a = 3$, $c = 2$, $e = 5$.

1. $3a - 2$ **2.** $4c + e$ **3.** $2c + 3a$ **4.** $5e - a$
5. $e - 2c$ **6.** $e - 2a$ **7.** $4c + 2e$ **8.** $7a - 5e$
9. $c - e$ **10.** $10a + c + e$ **11.** $a + c - e$ **12.** $a - c - e$

For questions **13** to **24**, $h = 3$, $m = -2$, $t = -3$.

13. $2m - 3$ **14.** $4t + 10$ **15.** $3h - 12$ **16.** $6m + 4$
17. $9t - 3$ **18.** $4h + 4$ **19.** $2m - 6$ **20.** $m + 2$
21. $3h + m$ **22.** $t - h$ **23.** $4m + 2h$ **24.** $3t - m$

For questions **25** to **36**, $x = -2$, $y = -1$, $k = 0$.

25. $3x + 1$ **26.** $2y + 5$ **27.** $6k + 4$ **28.** $3x + 2y$
29. $2k + x$ **30.** xy **31.** xk **32.** $2xy$
33 $2(x + k)$ **34.** $3(k + y)$ **35.** $5x - y$ **36.** $3k - 2x$

$2x^2$ means $2(x^2)$.
$(2x^2)$ means 'work out $2x$ and *then* square it'.
$-7x$ means $-7(x)$.
$-x^2$ means $-1(x^2)$.

Example

When $x = -2$, find the value of:
(a) $2x^2 - 5x$ (b) $(3x)^2 - x^2$

(a) $2x^2 - 5x = 2(-2)^2 - 5(-2)$
$= 2(4) + 10$
$= 18$

(b) $(3x)^2 - x^2 = (3 \times -2)^2 - 1(-2)^2$
$= (-6)^2 - 1(4)$
$= 36 - 4$
$= 32$

Exercise 6

If $x = -3$ and $y = 2$, evaluate the following:

1. x^2	**2.** $3x^2$	**3.** y^2	**4.** $4y^2$	**5.** $(2x)^2$
6. $2x^2$	**7.** $10 - x^2$	**8.** $10 - y^2$	**9.** $20 - 2x^2$	**10.** $20 - 3y^2$
11. $5 + 4x$	**12.** $x^2 - 2x$	**13.** $y^2 - 3x^2$	**14.** $x^2 - 3y$	**15.** $(2x)^2 - y^2$
16. $4x^2$	**17.** $(4x)^2$	**18.** $1 - x^2$	**19.** $y - x^2$	**20.** $x^2 + y^2$
21. $x^2 - y^2$	**22** $2 - 2x^2$	**23.** $(3x)^2 + 3$	**24.** $11 - xy$	**25.** $12 + xy$
26. $(2x)^2 - (3y)^2$	**27.** $2 - 3x^2$	**28.** $y^2 - x^2$	**29.** $x^2 + y^3$	**30.** $\frac{x}{y}$
31. $10 - 3x$	**32.** $2y^2$	**33.** $25 - 3y$	**34.** $(2y)^2$	**35.** $-7 + 3x$
36. $-8 + 10y$	**37.** $(xy)^2$	**38.** xy^2	**39.** $-7 + x^2$	**40.** $17 + xy$
41. $-5 - 2x^2$	**42.** $10 - (2x)^2$	**43.** $x^2 + 3x + 5$	**44.** $2x^2 - 4x + 1$	**45.** $\frac{x^2}{y}$

Example

When $a = -2$, $b = 3$, $c = -3$, evaluate:

(a) $\dfrac{2a(b^2 - a)}{c}$ (b) $\sqrt{(a^2 + b^2)}$

(a) $(b^2 - a) = 9 - (-2)$
$= 11$

$$\therefore \quad \frac{2a(b^2 - a)}{c} = \frac{2 \times (-2) \times (11)}{-3} = 14\tfrac{2}{3}$$

(b) $a^2 + b^2 = (-2)^2 + (3)^2$
$= 4 + 9$
$= 13$

$\sqrt{(a^2 + b^2)} = \sqrt{13}$

Exercise 7

Evaluate the following:

In questions **1** to **16**, $a = 4$, $b = -2$, $c = -3$.

1. $a(b + c)$	**2.** $a^2(b - c)$	**3.** $2c(a - c)$	**4.** $b^2(2a + 3c)$
5. $c^2(b - 2a)$	**6.** $2a^2(b + c)$	**7.** $2(a + b + c)$	**8.** $3c(a - b - c)$
9. $b^2 + 2b + a$	**10.** $c^2 - 3c + a$	**11.** $2b^2 - 3b$	**12.** $\sqrt{(a^2 + c^2)}$
13. $\sqrt{(ab + c^2)}$	**14.** $\sqrt{(c^2 - b^2)}$	**15.** $\frac{b^2}{a} + \frac{2c}{b}$	**16.** $\frac{c^2}{b} + \frac{4b}{a}$

In questions **17** to **32**, $k = -3$, $m = 1$, $n = -4$.

17. $k^2(2m - n)$	**18.** $5m\sqrt{(k^2 + n^2)}$	**19.** $\sqrt{(kn + 4m)}$	**20.** $kmn(k^2 + m^2 + n^2)$
21. $k^2m^2(m - n)$	**22.** $k^2 - 3k + 4$	**23.** $m^3 + m^2 + n^2 + n$	**24.** $k^3 + 3k$
25. $m(k^2 - n^2)$	**26.** $m\sqrt{(k - n)}$	**27.** $100k^2 + m$	**28.** $m^2(2k^2 - 3n^2)$
29. $\frac{2k + m}{k - n}$	**30.** $\frac{kn - k}{2m}$	**31.** $\frac{3k + 2m}{2n - 3k}$	**32.** $\frac{k + m + n}{k^2 + m^2 + n^2}$

In questions **33** to **48**, $w = -2$, $x = 3$, $y = 0$, $z = -\frac{1}{2}$.

33. $\frac{w}{z} + x$ **34.** $\frac{w + x}{z}$ **35.** $y\left(\frac{x + z}{w}\right)$ **36.** $x^2(z + wy)$

37. $x\sqrt{(x + wz)}$ **38.** $w^2\sqrt{(z^2 + y^2)}$ **39.** $2(w^2 + x^2 + y^2)$ **40.** $2x(w - z)$

41. $\frac{z}{w} + x$ **42.** $\frac{z + w}{x}$ **43.** $\frac{x + w}{z^2}$ **44.** $\frac{y^2 - w^2}{xz}$

45. $z^2 + 4z + 5$ **46.** $\frac{1}{w} + \frac{1}{z} + \frac{1}{x}$ **47.** $\frac{4}{z} + \frac{10}{w}$ **48.** $\frac{yz - xw}{xz - w}$

49. Find $K = \sqrt{\left(\frac{a^2 + b^2 + c^2 - 2c}{a^2 + b^2 + 4c}\right)}$ if $a = 3$, $b = -2$, $c = -1$.

50. Find $W = \frac{kmn(k + m + n)}{(k + m)(k + n)}$ if $k = \frac{1}{2}$, $m = -\frac{1}{3}$, $n = \frac{1}{4}$.

2.3 Brackets and simplifying

A term outside a bracket multiplies each of the terms inside the bracket.
This is the *distributive law*.

Example 1

$3(x - 2y) = 3x - 6y$

Example 2

$2x(x - 2y + z) = 2x^2 - 4xy + 2xz$

Example 3

$7y - 4(2x - 3) = 7y - 8x + 12$

In general,
numbers can be added to numbers
x's can be added to x's
y's can be added to y's
x^2's can be added to x^2's

But they must not be mixed.

Example 4

$2x + 3y + 3x^2 + 2y - x = x + 5y + 3x^2$

Example 5

$$\begin{aligned} 7x + 3x(2x - 3) &= 7x + 6x^2 - 9x \\ &= 6x^2 - 2x \end{aligned}$$

Exercise 8

Simplify as far as possible:

1. $3x + 4y + 7y$
2. $4a + 7b - 2a + b$
3. $3x - 2y + 4y$
4. $2x + 3x + 5$
5. $7 - 3x + 2 + 4x$
6. $5 - 3y - 6y - 2$
7. $5x + 2y - 4y - x^2$
8. $2x^2 + 3x + 5$
9. $2x - 7y - 2x - 3y$
10. $4a + 3a^2 - 2a$
11. $7a - 7a^2 + 7$
12. $x^2 + 3x^2 - 4x^2 + 5x$
13. $\frac{3}{a} + b + \frac{7}{a} - 2b$
14. $\frac{4}{x} - \frac{7}{y} + \frac{1}{x} + \frac{2}{y}$
15. $\frac{m}{x} + \frac{2m}{x}$
16. $\frac{5}{x} - \frac{7}{x} + \frac{1}{2}$
17. $\frac{3}{a} + b + \frac{2}{a} + 2b$
18. $\frac{n}{4} - \frac{m}{3} - \frac{n}{2} + \frac{m}{3}$
19. $x^3 + 7x^2 - 2x^3$
20. $(2x)^2 - 2x^2$
21. $(3y)^2 + x^2 - (2y)^2$
22. $(2x)^2 - (2y)^2 - (4x)^2$
23. $5x - 7x^2 - (2x)^2$
24. $\frac{3}{x^2} + \frac{5}{x^2}$

Remove the brackets and collect like terms:

25. $3x + 2(x + 1)$
26. $5x + 7(x - 1)$
27. $7 + 3(x - 1)$
28. $9 - 2(3x - 1)$
29. $3x - 4(2x + 5)$
30. $5x - 2x(x - 1)$
31. $7x + 3x(x - 4)$
32. $4(x - 1) - 3x$
33. $5x(x + 2) + 4x$
34. $3x(x - 1) - 7x^2$
35. $3a + 2(a + 4)$
36. $4a - 3(a - 3)$
37. $3ab - 2a(b - 2)$
38. $3y - y(2 - y)$
39. $3x - (x + 2)$
40. $7x - (x - 3)$
41. $5x - 2(2x + 2)$
42. $3(x - y) + 4(x + 2y)$
43. $x(x - 2) + 3x(x - 3)$
44. $3x(x + 4) - x(x - 2)$
45. $y(3y - 1) - (3y - 1)$
46. $7(2x + 2) - (2x + 2)$
47. $7b(a + 2) - a(3b + 3)$
48. $3(x - 2) - (x - 2)$

Two brackets

Example 1

$$(x + 5)(x + 3) = x(x + 3) + 5(x + 3)$$
$$= x^2 + 3x + 5x + 15$$
$$= x^2 + 8x + 15$$

Example 2

$$(2x - 3)(4y + 3) = 2x(4y + 3) - 3(4y + 3)$$
$$= 8xy + 6x - 12y - 9$$

Example 3

$$3(x + 1)(x - 2) = 3[x(x - 2) + 1(x - 2)]$$
$$= 3[x^2 - 2x + x - 2]$$
$$= 3x^2 - 3x - 6$$

Exercise 9

Remove the brackets and simplify:

1. $(x + 1)(x + 3)$
2. $(x + 3)(x + 2)$
3. $(y + 4)(y + 5)$
4. $(x - 3)(x + 4)$
5. $(x + 5)(x - 2)$
6. $(x - 3)(x - 2)$
7. $(a - 7)(a + 5)$
8. $(z + 9)(z - 2)$
9. $(x - 3)(x + 3)$
10. $(k - 11)(k + 11)$

11. $(2x+1)(x-3)$ **12.** $(3x+4)(x-2)$ **13.** $(2y-3)(y+1)$ **14.** $(7y-1)(7y+1)$
15. $(3x-2)(3x+2)$ **16.** $(3a+b)(2a+b)$ **17.** $(3x+y)(x+2y)$ **18.** $(2b+c)(3b-c)$
19. $(5x-y)(3y-x)$ **20.** $(3b-a)(2a+5b)$ **21.** $2(x-1)(x+2)$ **22.** $3(x-1)(2x+3)$
23. $4(2y-1)(3y+2)$ **24.** $2(3x+1)(x-2)$ **25.** $4(a+2b)(a-2b)$ **26.** $x(x-1)(x-2)$
27. $2x(2x-1)(2x+1)$ **28.** $3y(y-2)(y+3)$ **29.** $x(x+y)(x+z)$ **30.** $3z(a+2m)(a-m)$

Be careful with an expression like $(x-3)^2$. It is not x^2-9 or even x^2+9.

$$\begin{aligned}(x-3)^2 &= (x-3)(x-3)\\ &= x(x-3)-3(x-3)\\ &= x^2-6x+9\end{aligned}$$

Another common mistake occurs with an expression like $4-(x-1)^2$.

$$\begin{aligned}4-(x-1)^2 &= 4-1(x-1)(x-1)\\ &= 4-1(x^2-2x+1)\\ &= 4-x^2+2x-1\\ &= 3+2x-x^2\end{aligned}$$

Exercise 10

Remove the brackets and simplify:

1. $(x+4)^2$ **2.** $(x+2)^2$ **3.** $(x-2)^2$ **4.** $(2x+1)^2$
5. $(y-5)^2$ **6.** $(3y+1)^2$ **7.** $(x+y)^2$ **8.** $(2x+y)^2$
9. $(a-b)^2$ **10.** $(2a-3b)^2$ **11.** $3(x+2)^2$ **12.** $(3-x)^2$
13. $(3x+2)^2$ **14.** $(a-2b)^2$ **15.** $(x+1)^2+(x+2)^2$ **16.** $(x-2)^2+(x+3)^2$
17. $(x+2)^2+(2x+1)^2$ **18.** $(y-3)^2+(y-4)^2$ **19.** $(x+2)^2-(x-3)^2$ **20.** $(x-3)^2-(x+1)^2$
21. $(y-3)^2-(y+2)^2$ **22.** $(2x+1)^2-(x+3)^2$ **23.** $3(x+2)^2-(x+4)^2$ **24.** $2(x-3)^2-3(x+1)^2$

2.4 Linear equations

- If the x term is negative, take it to the other side, where it becomes positive.

Example 1

$$\begin{aligned}4-3x &= 2\\ 4 &= 2+3x\\ 2 &= 3x\\ \frac{2}{3} &= x\end{aligned}$$

- If there are x terms on both sides, collect them on one side.

Example 2

$$\begin{aligned}2x-7 &= 5-3x\\ 2x+3x &= 5+7\\ 5x &= 12\\ x &= \frac{12}{5} = 2\tfrac{2}{5}\end{aligned}$$

- If there is a fraction in the x term, multiply out to simplify the equation.

Example 3

$$\frac{2x}{3} = 10$$
$$2x = 30$$
$$x = \frac{30}{2} = 15$$

Exercise 11

Solve the following equations:

1. $2x - 5 = 11$
2. $3x - 7 = 20$
3. $2x + 6 = 20$
4. $5x + 10 = 60$
5. $8 = 7 + 3x$
6. $12 = 2x - 8$
7. $-7 = 2x - 10$
8. $3x - 7 = -10$
9. $12 = 15 + 2x$
10. $5 + 6x = 7$
11. $\frac{x}{5} = 7$
12. $\frac{x}{10} = 13$
13. $7 = \frac{x}{2}$
14. $\frac{x}{2} = \frac{1}{3}$
15. $\frac{3x}{2} = 5$
16. $\frac{4x}{5} = -2$
17. $7 = \frac{7x}{3}$
18. $\frac{3}{4} = \frac{2x}{3}$
19. $\frac{5x}{6} = \frac{1}{4}$
20. $-\frac{3}{4} = \frac{3x}{5}$
21. $\frac{x}{2} + 7 = 12$
22. $\frac{x}{3} - 7 = 2$
23. $\frac{x}{5} - 6 = -2$
24. $4 = \frac{x}{2} - 5$
25. $10 = 3 + \frac{x}{4}$
26. $\frac{a}{5} - 1 = -4$
27. $100x - 1 = 98$
28. $7 = 7 + 7x$
29. $\frac{x}{100} + 10 = 20$
30. $1000x - 5 = -6$
31. $-4 = -7 + 3x$
32. $2x + 4 = x - 3$
33. $x - 3 = 3x + 7$
34. $5x - 4 = 3 - x$
35. $4 - 3x = 1$
36. $5 - 4x = -3$
37. $7 = 2 - x$
38. $3 - 2x = x + 12$
39. $6 + 2a = 3$
40. $a - 3 = 3a - 7$
41. $2y - 1 = 4 - 3y$
42. $7 - 2x = 2x - 7$
43. $7 - 3x = 5 - 2x$
44. $8 - 2y = 5 - 5y$
45. $x - 16 = 16 - 2x$
46. $x + 2 = 3{\cdot}1$
47. $-x - 4 = -3$
48. $-3 - x = -5$
49. $-\frac{x}{2} + 1 = -\frac{1}{4}$
50. $-\frac{3}{5} + \frac{x}{10} = -\frac{1}{5} - \frac{x}{5}$

Example

$$x - 2(x - 1) = 1 - 4(x + 1)$$
$$x - 2x + 2 = 1 - 4x - 4$$
$$x - 2x + 4x = 1 - 4 - 2$$
$$3x = -5$$
$$x = -\frac{5}{3}$$

Exercise 12

Solve the following equations:

1. $x + 3(x + 1) = 2x$
2. $1 + 3(x - 1) = 4$
3. $2x - 2(x + 1) = 5x$
4. $2(3x - 1) = 3(x - 1)$
5. $4(x - 1) = 2(3 - x)$
6. $4(x - 1) - 2 = 3x$
7. $4(1 - 2x) = 3(2 - x)$
8. $3 - 2(2x + 1) = x + 17$
9. $4x = x - (x - 2)$
10. $7x = 3x - (x + 20)$
11. $5x - 3(x - 1) = 39$
12. $3x + 2(x - 5) = 15$
13. $7 - (x + 1) = 9 - (2x - 1)$
14. $10x - (2x + 3) = 21$
15. $3(2x + 1) + 2(x - 1) = 23$
16. $5(1 - 2x) - 3(4 + 4x) = 0$
17. $7x - (2 - x) = 0$
18. $3(x + 1) = 4 - (x - 3)$
19. $3y + 7 + 3(y - 1) = 2(2y + 6)$
20. $4(y - 1) + 3(y + 2) = 5(y - 4)$
21. $4x - 2(x + 1) = 5(x + 3) + 5$
22. $7 - 2(x - 1) = 3(2x - 1) + 2$
23. $10(2x + 3) - 8(3x - 5) + 5(2x - 8) = 0$
24. $2(x + 4) + 3(x - 10) = 8$
25. $7(2x - 4) + 3(5 - 3x) = 2$
26. $10(x + 4) - 9(x - 3) - 1 = 8(x + 3)$
27. $5(2x - 1) - 2(x - 2) = 7 + 4x$
28. $6(3x - 4) - 10(x - 3) = 10(2x - 3)$
29. $3(x - 3) - 7(2x - 8) - (x - 1) = 0$
30. $5 + 2(x + 5) = 10 - (4 - 5x)$
31. $6x + 30(x - 12) = 2(x - 1\frac{1}{2})$
32. $3(2x - \frac{2}{3}) - 7(x - 1) = 0$
33. $5(x - 1) + 17(x - 2) = 2x + 1$
34. $6(2x - 1) + 9(x + 1) = 8(x - 1\frac{1}{4})$
35. $7(x + 4) - 5(x + 3) + (4 - x) = 0$
36. $0 = 9(3x + 7) - 5(x + 2) - (2x - 5)$
37. $10(2{\cdot}3 - x) - 0{\cdot}1(5x - 30) = 0$
38. $8(2\frac{1}{2}x - \frac{3}{4}) - \frac{1}{4}(1 - x) = \frac{1}{2}$
39. $(6 - x) - (x - 5) - (4 - x) = -\dfrac{x}{2}$
40. $10\left(1 - \dfrac{x}{10}\right) - (10 - x) - \dfrac{1}{100}(10 - x) = 0{\cdot}05$

Example

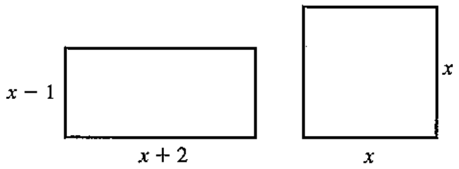
$$(x + 3)^2 = (x + 2)^2 + 3^2$$
$$(x + 3)(x + 3) = (x + 2)(x + 2) + 9$$
$$x^2 + 6x + 9 = x^2 + 4x + 4 + 9$$
$$6x + 9 = 4x + 13$$
$$2x = 4$$
$$x = 2$$

Exercise 13

Solve the following equations:

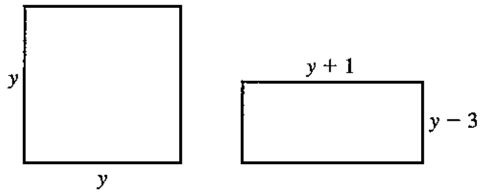
1. $x^2 + 4 = (x + 1)(x + 3)$
2. $x^2 + 3x = (x + 3)(x + 1)$
3. $(x + 3)(x - 1) = x^2 + 5$
4. $(x + 1)(x + 4) = (x - 7)(x + 6)$
5. $(x - 2)(x + 3) = (x - 7)(x + 7)$
6. $(x - 5)(x + 4) = (x + 7)(x - 6)$
7. $2x^2 + 3x = (2x - 1)(x + 1)$
8. $(2x - 1)(x - 3) = (2x - 3)(x - 1)$
9. $x^2 + (x + 1)^2 = (2x - 1)(x + 4)$
10. $x(2x + 6) = 2(x^2 - 5)$
11. $(x + 1)(x - 3) + (x + 1)^2 = 2x(x - 4)$
12. $(2x + 1)(x - 4) + (x - 2)^2 = 3x(x + 2)$
13. $(x + 2)^2 - (x - 3)^2 = 3x - 11$
14. $x(x - 1) = 2(x - 1)(x + 5) - (x - 4)^2$
15. $(2x + 1)^2 - 4(x - 3)^2 = 5x + 10$
16. $2(x + 1)^2 - (x - 2)^2 = x(x - 3)$

17. The area of the rectangle shown exceeds the area of the square by $2\,cm^2$. Find x.

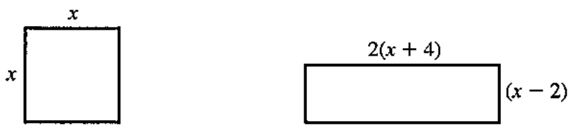


18. The area of the square exceeds the area of the rectangle by $13\,m^2$. Find y.

Remember
For a rectangle:
Area = length × width



19. The area of the square is half the area of the rectangle. Find x.



When solving equations involving fractions, multiply both sides of the equation by a suitable number to eliminate the fractions.

Example 1

$\frac{5}{x} = 2$

$5 = 2x$ (multiply both sides by x)

$\frac{5}{2} = x$

Example 2

$$\frac{x+4}{4} = \frac{2x-1}{3} \qquad \ldots(A)$$

$$12\frac{(x+3)}{4} = 12\frac{(2x-1)}{3}$$

(multiply both sides by 12)

$$\therefore \quad 3(x+3) = 4(2x-1) \qquad \ldots(B)$$

$$3x+9 = 8x-4$$

$$13 = 5x$$

$$\frac{13}{5} = x$$

$$x = 2\tfrac{3}{5}$$

Note: It is possible to go straight from line (A) to line (B) by 'cross-multiplying'.

Example 3

$$\frac{5}{(x-1)} + 2 = 12$$

$$\frac{5}{(x-1)} = 10$$

$$5 = 10(x-1)$$

$$5 = 10x - 10$$

$$15 = 10x$$

$$\frac{15}{10} = x$$

$$x = 1\tfrac{1}{2}$$

Exercise 14

Solve the following equations:

1. $\frac{7}{x} = 21$
2. $30 = \frac{6}{x}$
3. $\frac{5}{x} = 3$
4. $\frac{9}{x} = -3$
5. $11 = \frac{5}{x}$
6. $-2 = \frac{4}{x}$
7. $\frac{x}{4} = \frac{3}{2}$
8. $\frac{x}{3} = 1\frac{1}{4}$
9. $\frac{x+1}{3} = \frac{x-1}{4}$
10. $\frac{x+3}{2} = \frac{x-4}{5}$
11. $\frac{2x-1}{3} = \frac{x}{2}$
12. $\frac{3x+1}{5} = \frac{2x}{3}$
13. $\frac{8-x}{2} = \frac{2x+2}{5}$
14. $\frac{x+2}{7} = \frac{3x+6}{5}$
15. $\frac{1-x}{2} = \frac{3-x}{3}$
16. $\frac{2}{x-1} = 1$
17. $\frac{x}{3} + \frac{x}{4} = 1$
18. $\frac{x}{3} + \frac{x}{2} = 4$
19. $\frac{x}{2} - \frac{x}{5} = 3$
20. $\frac{x}{3} = 2 + \frac{x}{4}$
21. $\frac{5}{x-1} = \frac{10}{x}$

22. $\frac{12}{2x-3}=4$

23. $2=\frac{18}{x+4}$

24. $\frac{5}{x+5}=\frac{15}{x+7}$

25. $\frac{9}{x}=\frac{5}{x-3}$

26. $\frac{4}{x-1}=\frac{10}{3x-1}$

27. $\frac{-7}{x-1}=\frac{14}{5x+2}$

28. $\frac{4}{x+1}=\frac{7}{3x-2}$

29. $\frac{x+1}{2}+\frac{x-1}{3}=\frac{1}{6}$

30. $\frac{1}{3}(x+2)=\frac{1}{5}(3x+2)$

31. $\frac{1}{2}(x-1)-\frac{1}{6}(x+1)=0$

32. $\frac{1}{4}(x+5)-\frac{2x}{3}=0$

33. $\frac{4}{x}+2=3$

34. $\frac{6}{x}-3=7$

35. $\frac{9}{x}-7=1$

36. $-2=1+\frac{3}{x}$

37. $4-\frac{4}{x}=0$

38. $5-\frac{6}{x}=-1$

39. $7-\frac{3}{2x}=1$

40. $4+\frac{5}{3x}=-1$

41. $\frac{9}{2x}-5=0$

42. $\frac{x-1}{5}-\frac{x-1}{3}=0$

43. $\frac{x-1}{4}-\frac{2x-3}{5}=\frac{1}{20}$

44. $\frac{4}{1-x}=\frac{3}{1+x}$

45. $\frac{x+1}{4}-\frac{x}{3}=\frac{1}{12}$

46. $\frac{2x+1}{8}-\frac{x-1}{3}=\frac{5}{24}$

2.5 Problems solved by linear equations

- Let the unknown quantity be x (or any other letter) and state the units (where appropriate).
- Express the given statement in the form of an equation.
- Solve the equation for x and give the answer in *words*. (Do not finish by writing '$x = 3$'.)
- Check your solution using the problem (not your equation).

Example 1

The sum of three consecutive whole numbers is 78. Find the numbers.

(a) Let the smallest number be x; then the other numbers are $(x + 1)$ and $(x + 2)$.

(b) Form an equation:
$x+(x+1)+(x+2)=78$

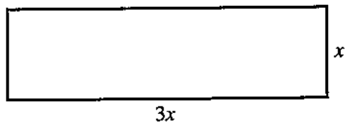
(c) Solve: $3x = 75$
$x = 25$
In words:
The three numbers are 25, 26 and 27.

(d) Check: $25 + 26 + 27 = 78$

Example 2

The length of a rectangle is three times the width. If the perimeter is 36 cm, find the width.

(a) Let the width of the rectangle be x cm.
Then the length of the rectangle is $3x$ cm.



(b) Form an equation.
$x + 3x + x + 3x = 36$

(c) Solve: $8x = 36$

$$x = \frac{36}{8}$$

$$x = 4{\cdot}5$$

In words:
The width of the rectangle is 4·5 cm.

(d) Check: If width = 4·5 cm
length = 13·5 cm
perimeter = 36 cm

Exercise 15

Solve each problem by forming an equation. The first questions are easy but should still be solved using an equation, in order to practise the method:

1. The sum of three consecutive numbers is 276. Find the numbers.

2. The sum of four consecutive numbers is 90. Find the numbers.

3. The sum of three consecutive odd numbers is 177. Find the numbers.

4. Find three consecutive even numbers which add up to 1524.

5. When a number is doubled and then added to 13, the result is 38. Find the number.

6. When a number is doubled and then added to 24, the result is 49. Find the number.

7. When 7 is subtracted from three times a certain number, the result is 28. What is the number?

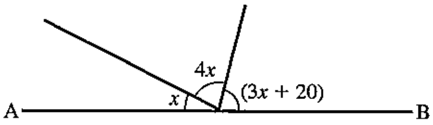
8. The sum of two numbers is 50. The second number is five times the first. Find the numbers.

9. Two numbers are in the ratio 1 : 11 and their sum is 15. Find the numbers.

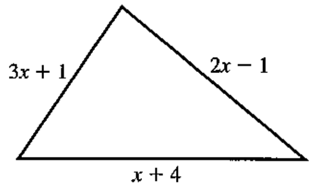
10. The length of a rectangle is twice the width.
If the perimeter is 20 cm, find the width.

11. The width of a rectangle is one third of the length. If the perimeter is 96 cm, find the width.

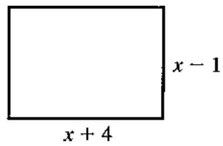
12. If AB is a straight line, find x.
(The angles on a straight line add to 180°.)



13. If the perimeter of the triangle is 22 cm, find the length of the shortest side.



14. If the perimeter of the rectangle is 34 cm, find x.



15. The difference between two numbers is 9.
Find the numbers, if their sum is 46.

16. The three angles in a triangle are in the ratio $1:3:5$. Find them.

Angles in a triangle add to 180°.

17. The three angles in a triangle are in the ratio $3:4:5$. Find them.

18. The product of two consecutive odd numbers is 10 more than the square of the smaller number. Find the smaller number.

19. The product of two consecutive even numbers is 12 more than the square of the smaller number. Find the numbers.

20. The sum of three numbers is 66. The second number is twice the first and six less than the third. Find the numbers.

21. The sum of three numbers is 28. The second number is three times the first and the third is 7 less than the second. What are the numbers?

22. David weighs 5 kg less than John, who in turn is 8 kg lighter than Paul. If their total weight is 197 kg, how heavy is each person?

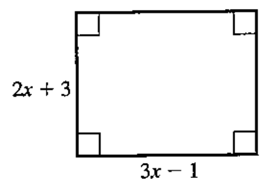
23. Brian is 2 years older than Bob who is 7 years older than Mark. If their combined age is 61 years, find the age of each person.

24. Richard has four times as many marbles as John. If Richard gave 18 to John they would have the same number. How many marbles has each?

25. Stella has five times as many books as Tina.
If Stella gave 16 books to Tina, they would each have the same number. How many books did each girl have?

26. The result of trebling a number is the same as adding 12 to it. What is the number?

27. Find the area of the rectangle if the perimeter is 52 cm.
(The perimeter is the distance around the edge of the rectangle.)



28. The result of trebling a number and subtracting 5 is the same as doubling the number and adding 9. What is the number?

29. Two girls have \$76 between them. If the first gave the second \$7 they would each have the same amount of money. How much did each girl have?

30. A tennis racket costs \$12 more than a hockey stick. If the price of the two is \$31, find the cost of the tennis racket.

Example

A man goes out at 16:42 and arrives at a post box, 6 km away, at 17:30. He walked part of the way at 5 km/h and then, realising the time, he ran the rest of the way at 10 km/h. How far did he have to run?

- Let the distance he ran be x km. Then the distance he walked $= (6 - x)$ km.
- Time taken to walk $(6 - x)$ km at 5 km/h $= \frac{(6-x)}{5}$ hours.

  Time taken to run x km at 10 km/h $= \frac{x}{10}$ hours.

  $$\text{Total time taken} = 48 \text{ minutes} = \frac{4}{5} \text{ hour}$$

  $$\therefore \frac{(6-x)}{5} + \frac{x}{10} = \frac{4}{5}$$

- Multiply by 10:
  $$2(6 - x) + x = 8$$
  $$12 - 2x + x = 8$$
  $$4 = x$$
  He ran a distance of 4 km.

- Check:

  Time to run 4 km $= \frac{4}{10} = \frac{2}{5}$ hour.

  Time to walk 2 km $= \frac{2}{5}$ hour.

  $$\text{Total time taken} = \left(\frac{2}{5} + \frac{2}{5}\right) \text{h} = \frac{4}{5} \text{h}$$

Exercise 16

1. Every year a man is paid \$500 more than the previous year. If he receives \$17 800 over four years, what was he paid in the first year?

2. A man buys x cans of beer at 30 cents each and $(x + 4)$ cans of lager at 35 cents each. The total cost was \$3·35. Find x.

3. The length of a straight line ABC is 5 m.
If AB : BC = 2 : 5, find the length of AB.

4. The opposite angles of a cyclic quadrilateral are $(3x + 10)°$ and $(2x + 20)°$. Find the angles.

Opposite angles of a cyclic quadrilateral add to 180°.

5. The interior angles of a hexagon are in the ratio 1 : 2 : 3 : 4 : 5 : 9. Find the angles. This is an example of a concave hexagon. Try to sketch the hexagon.

Interior angles of a hexagon add to 720°.

6. A man is 32 years older than his son. Ten years ago he was three times as old as his son was then. Find the present age of each.

7. A man runs to a telephone and back in 15 minutes. His speed on the way to the telephone is 5 m/s and his speed on the way back is 4 m/s. Find the distance to the telephone.

8. A car completes a journey in 10 minutes. For the first half of the distance the speed was 60 km/h and for the second half the speed was 40 km/h. How far is the journey?

9. A lemming runs from a point A to a cliff at 4 m/s, jumps over the edge at B and falls to C at an average speed of 25 m/s. If the total distance from A to C is 500 m and the time taken for the journey is 41 seconds, find the height BC of the cliff.

10. A bus is travelling with 48 passengers. When it arrives at a stop, x passengers get off and 3 get on. At the next stop half the passengers get off and 7 get on. There are now 22 passengers. Find x.

11. A bus is travelling with 52 passengers. When it arrives at a stop, y passengers get off and 4 get on. At the next stop one-third of the passengers get off and 3 get on. There are now 25 passengers. Find y.

12. Mr Lee left his fortune to his 3 sons, 4 daughters and his wife. Each son received twice as much as each daughter and his wife received \$6000, which was a quarter of the money. How much did each son receive?

13. In a regular polygon with n sides each interior angle is $180 - \dfrac{360}{n}$ degrees. How many sides does a polygon have if each interior angle is 156°?

14. A sparrow flies to see a friend at a speed of 4 km/h. His friend is out, so the sparrow immediately returns home at a speed of 5 km/h. The complete journey took 54 minutes. How far away does his friend live?

15. Consider the equation $an^2 = 182$ where a is any number between 2 and 5 and n is a positive integer. What are the possible values of n?

16. Consider the equation $\frac{k}{x} = 12$ where k is any number between 20 and 65 and x is a positive integer. What are the possible values of x?

2.6 Simultaneous equations

To find the value of two unknowns in a problem, *two* different equations must be given that relate the unknowns to each other. These two equations are called *simultaneous* equations.

Substitution method

This method is used when one equation contains a unit quantity of one of the unknowns, as in equation [2] of the example below.

Example

$3x - 2y = 0$ ...[1]
$2x + y = 7$ ...[2]

(a) Label the equations so that the working is made clear.
(b) In *this* case, write y in terms of x from equation [2].
(c) Substitute this expression for y in equation [1] and solve to find x.
(d) Find y from equation [2] using this value of x.

$2x + y = 7$ ...[2]
$y = 7 - 2x$

Substituting in [1]
$3x - 2(7 - 2x) = 0$
$3x - 14 + 4x = 0$
$7x = 14$
$x = 2$

Substituting in [2]
$2 \times 2 + y = 7$
$y = 3$

The solutions are $x = 2$, $y = 3$.

These values of x and y are the only pair which simultaneously satisfy *both* equations.

Exercise 17

Use the substitution method to solve the following:

1. $2x + y = 5$
$x + 3y = 5$

2. $x + 2y = 8$
$2x + 3y = 14$

3. $3x + y = 10$
$x - y = 2$

4. $2x + y = -3$
$x - y = -3$

5. $4x + y = 14$
$x + 5y = 13$

6. $x + 2y = 1$
$2x + 3y = 4$

7. $2x + y = 5$
$3x - 2y = 4$

8. $2x + y = 13$
$5x - 4y = 13$

9. $7x + 2y = 19$
$x - y = 4$

10. $b - a = -5$
$a + b = -1$

11. $a + 4b = 6$
$8b - a = -3$

12. $a + b = 4$
$2a + b = 5$

13. $3m = 2n - 6\frac{1}{2}$
$4m + n = 6$

14. $2w + 3x - 13 = 0$
$x + 5w - 13 = 0$

15. $x + 2(y - 6) = 0$
$3x + 4y = 30$

16. $2x = 4 + z$
$6x - 5z = 18$

17. $3m - n = 5$
$2m + 5n = 7$

18. $5c - d - 11 = 0$
$4d + 3c = -5$

It is useful, at this point to revise the operations of addition and subtraction with negative numbers.

Example

Simplify:

(a) $-7 + -4 = -7 - 4 = -11$

(b) $-3x + (-4x) = -3x - 4x = -7x$

(c) $4y - (-3y) = 4y + 3y = 7y$

(d) $3a + (-3a) = 3a - 3a = 0$

Exercise 18

Evaluate:

1. $7 + (-6)$
2. $8 + (-11)$
3. $5 - (+7)$
4. $6 - (-9)$
5. $-8 + (-4)$
6. $-7 - (-4)$
7. $10 + (-12)$
8. $-7 - (+4)$
9. $-10 - (+11)$
10. $-3 - (-4)$
11. $4 - (+4)$
12. $8 - (-7)$
13. $-5 - (+5)$
14. $-7 - (-10)$
15. $16 - (+10)$
16. $-7 - (+4)$
17. $-6 - (-8)$
18. $10 - (+5)$
19. $-12 + (-7)$
20. $7 + (-11)$

Simplify:

21. $3x + (-2x)$
22. $4x + (-7x)$
23. $6x - (+2x)$
24. $10y - (+6y)$
25. $6y - (-3y)$
26. $7x + (-4x)$
27. $-5x + (-3x)$
28. $-3x - (-7x)$
29. $5x - (+3x)$
30. $-7y - (-10y)$

Elimination method

Use this method when the first method is unsuitable (some prefer to use it for every question).

Example 1

$x + 2y = 8$ ...[1]
$2x + 3y = 14$ ...[2]

(a) Label the equations so that the working is made clear.
(b) Choose an unknown in one of the equations and multiply the equations by a factor or factors so that this unknown has the same coefficient in both equations.
(c) Eliminate this unknown from the two equations by subtracting them, then solve for the remaining unknown.
(d) Substitute in the first equation and solve for the eliminated unknown.

$x + 2y = 8$ ...[1]

[1] × 2 $2x + 4y = 16$ ...[3]
$2x + 3y = 14$ ...[2]

Subtract [2] from [3]
$y = 2$

Substituting in [1]
$x + 2 \times 2 = 8$
$x = 8 - 4$
$x = 4$

The solutions are $x = 4$, $y = 2$.

Example 2

$2x + 3y = 5$ ...[1]
$5x - 2y = -16$ ...[2]

[1] × 5 $10x + 15y = 25$ ...[3]
[2] × 2 $10x - 4y = -32$ ...[4]

[3] − 4 $15y - (-4y) = 25 - (-32)$
$19y = 57$
$y = 3$

Substitute in [1]
$2x + 3 \times 3 = 5$
$2x = 5 - 9 = -4$
$x = -2$

The solutions are $x = -2$, $y = 3$.

Exercise 19

Use the elimination method to solve the following:

1. $2x+5y=24$
$4x+3y=20$

2. $5x+2y=13$
$2x+6y=26$

3. $3x+y=11$
$9x+2y=28$

4. $x+2y=17$
$8x+3y=45$

5. $3x+2y=19$
$x+8y=21$

6. $2a+3b=9$
$4a+b=13$

7. $2x+3y=11$
$3x+4y=15$

8. $3x+8y=27$
$4x+3y=13$

9. $2x+7y=17$
$5x+3y=-1$

10. $5x+3y=23$
$2x+4y=12$

11. $7x+5y=32$
$3x+4y=23$

12. $3x+2y=4$
$4x+5y=10$

13. $3x+2y=11$
$2x-y=-3$

14. $3x+2y=7$
$2x-3y=-4$

15. $x+2y=-4$
$3x-y=9$

16. $5x-7y=27$
$3x-4y=16$

17. $3x-2y=7$
$4x+y=13$

18. $x-y=-1$
$2x-y=0$

19. $y-x=-1$
$3x-y=5$

20. $x-3y=-5$
$2y+3x+4=0$

21. $x+3y-7=0$
$2y-x-3=0$

22. $3a-b=9$
$2a+2b=14$

23. $3x-y=9$
$4x-y=-14$

24. $x+2y=4$
$3x+y=9\frac{1}{2}$

25. $2x-y=5$
$\frac{x}{4}+\frac{y}{3}=2$

26. $3x-y=17$
$\frac{x}{5}+\frac{y}{2}=0$

27. $3x-2y=5$
$\frac{2x}{3}+\frac{y}{2}=-\frac{7}{9}$

28. $2x=11-y$
$\frac{x}{5}-\frac{y}{4}=1$

29. $4x-0{\cdot}5y=12{\cdot}5$
$3x+0{\cdot}8y=8{\cdot}2$

30. $0{\cdot}4x+3y=2{\cdot}6$
$x-2y=4{\cdot}6$

2.7 Problems solved by simultaneous equations

Example

A motorist buys 24 litres of petrol and 5 litres of oil for \$10·70, while another motorist buys 18 litres of petrol and 10 litres of oil for \$12·40. Find the cost of 1 litre of petrol and 1 litre of oil at this garage.

Let cost of 1 litre of petrol be x cents.
Let cost of 1 litre of oil be y cents.

We have, $24x+5y=1070$ ...[1]
$18x+10y=1240$ ...[2]

(a) Multiply [1] by 2,
$48x+10y=2140$ ...[3]

(b) Subtract [2] from [3],
$30x=900$
$x=30$

(c) Substitute $x=30$ into equation [2]
$18(30)+10y=1240$
$10y=1240-540$
$10y=700$
$y=70$

1 litre of petrol costs 30 cents and 1 litre of oil costs 70 cents.

Exercise 20

Solve each problem by forming a pair of simultaneous equations:

1. Find two numbers with a sum of 15 and a difference of 4.
2. Twice one number added to three times another gives 21. Find the numbers, if the difference between them is 3.
3. The average of two numbers is 7, and three times the difference between them is 18. Find the numbers.
4. The line, with equation $y + ax = c$, passes through the points (1, 5) and (3, 1). Find a and c.
   Hint: For the point (1, 5) put $x = 1$ and $y = 5$ into $y + ax = c$, etc.
5. The line $y = mx + c$ passes through (2, 5) and (4, 13). Find m and c.
6. The curve $y = ax^2 + bx$ passes through (2, 0) and (4, 8). Find a and b.
7. A fishing enthusiast buys fifty maggots and twenty worms for \$1·10 and her mother buys thirty maggots and forty worms for \$1·50. Find the cost of one maggot and one worm.
8. A television addict can buy either two televisions and three video-recorders for \$1750 or four televisions and one video-recorder for \$1250. Find the cost of one of each.
9. Half the difference between two numbers is 2. The sum of the greater number and twice the smaller number is 13. Find the numbers.
10. A snake can lay either white or brown eggs. Three white eggs and two brown eggs weigh 13 grams, while five white eggs and four brown eggs weigh 24 grams. Find the weight of a brown egg and of a white egg.
11. A tortoise makes a journey in two parts; it can either walk at 4 cm/s or crawl at 3 cm/s. If the tortoise walks the first part and crawls the second, it takes 110 seconds. If it crawls the first part and walks the second, it takes 100 seconds. Find the lengths of the two parts of the journey.
12. A cyclist completes a journey of 500 m in 22 seconds, part of the way at 10 m/s and the remainder at 50 m/s. How far does she travel at each speed?
13. A bag contains forty coins, all of them either 2 cent or 5 cent coins. If the value of the money in the bag is \$1.55, find the number of each kind.

> **Remember**
> 100 cents = \$1

14. A slot machine takes only 10 cent and 50 cent coins and contains a total of twenty-one coins altogether. If the value of the coins is \$4·90, find the number of coins of each value.
15. Thirty tickets were sold for a concert, some at 60 cents and the rest at \$1. If the total raised was \$22, how many had the cheaper tickets?

16. The wage bill for five men and six women workers is \$6700, while the bill for eight men and three women is \$6100. Find the wage for a man and for a woman.

17. A fish can swim at 14 m/s with the current and at 6 m/s against it. Find the speed of the current and the speed of the fish in still water.

18. If the numerator and denominator of a fraction are both decreased by one the fraction becomes $\frac{2}{3}$. If the numerator and denominator are both increased by one the fraction becomes $\frac{3}{4}$. Find the original fraction.

19. The denominator of a fraction is 2 more than the numerator. If both denominator and numerator are increased by 1 the fraction becomes $\frac{2}{3}$. Find the original fraction.

20. In three years' time a pet mouse will be as old as his owner was four years ago. Their present ages total 13 years. Find the age of each now.

21. Find two numbers where three times the smaller number exceeds the larger by 5 and the sum of the numbers is 11.

22. A straight line passes through the points (2, 4) and (−1, −5). Find its equation.

23. A spider can walk at a certain speed and run at another speed. If she walks for 10 seconds and runs for 9 seconds she travels 85 m. If she walks for 30 seconds and runs for 2 seconds she travels 130 m. Find her speeds of walking and running.

24. A wallet containing \$40 has three times as many \$1 notes as \$5 notes. Find the number of each kind.

25. At the present time a man is four times as old as his son. Six years ago he was 10 times as old. Find their present ages.

26. A submarine can travel at 25 knots with the wind and at 16 knots against it. Find the speed of the wind and the speed of the submarine in still air.

27. The curve $y = ax^2 + bx + c$ passes through the points (1, 8), (0, 5) and (3, 20). Find the values of a, b and c and hence the equation of the curve.

28. The curve $y = ax^2 + bx + c$ passes through the points (1, 4), (−2, 19) and (0, 5). Find the equation of the curve.

29. The curve $y = ax^2 + bx + c$ passes through (1, 8), (−1, 2) and (2, 14). Find the equation of the curve.

30. The curve $y = ax^2 + bx + c$ passes through (2, 5), (3, 12) and (−1, −4). Find the equation of the curve.

2.8 Factorising

Earlier in this section we expanded expressions such as $x(3x-1)$ to give $3x^2 - x$.
The reverse of this process is called *factorising*.

Example

Factorise: (a) $x^2 + 7x$ (b) $3y^2 - 12y$ (c) $6a^2b - 10ab^2$

(a) x is common to x^2 and $7x$.
$\therefore \quad x^2 + 7x = x(x+7)$
The factors are x and $(x+7)$.

(b) $3y$ is common.
$\therefore \quad 3y^2 - 12y = 3y(y-4)$

(c) $2ab$ is common.
$\therefore \quad 6a^2b - 10ab^2 = 2ab(3a - 5b)$

Exercise 21

Factorise the following expressions completely:

1. $x^2 + 5x$	**2.** $x^2 - 6x$	**3.** $7x - x^2$	**4.** $y^2 + 8y$
5. $2y^2 + 3y$	**6.** $6y^2 - 4y$	**7.** $3x^2 - 21x$	**8.** $16a - 2a^2$
9. $6c^2 - 21c$	**10.** $15x - 9x^2$	**11.** $56y - 21y^2$	**12.** $ax + bx + 2cx$
13. $x^2 + xy + 3xz$	**14.** $x^2y + y^3 + z^2y$	**15.** $3a^2b + 2ab^2$	**16.** $x^2y + xy^2$
17. $6a^2 + 4ab + 2ac$	**18.** $ma + 2bm + m^2$	**19.** $2kx + 6ky + 4kz$	**20.** $ax^2 + ay + 2ab$
21. $x^2k + xk^2$	**22.** $a^3b + 2ab^2$	**23.** $abc - 3b^2c$	**24.** $2a^2e - 5ae^2$
25. $a^3b + ab^3$	**26.** $x^3y + x^2y^2$	**27.** $6xy^2 - 4x^2y$	**28.** $3ab^3 - 3a^3b$
29. $2a^3b + 5a^2b^2$	**30.** $ax^2y - 2ax^2z$	**31.** $2abx + 2ab^2 + 2a^2b$	**32.** $ayx + yx^3 - 2y^2x^2$

Example 1

Factorise $ah + ak + bh + bk$.

(a) Divide into pairs, $ah + ak \mid + bh + bk$.

(b) a is common to the first pair
b is common to the second pair
$a(h+k) + b(h+k)$

(c) $(h+k)$ is common to both terms.
Thus we have $(h+k)(a+b)$

Example 2

Factorise $6mx - 3nx + 2my - ny$.

(a) $6mx - 3nx \mid + 2my - ny$

(b) $= 3x(2m-n) + y(2m-n)$

(c) $= (2m-n)(3x+y)$

Exercise 22

Factorise the following expressions:

1. $ax + ay + bx + by$
2. $ay + az + by + bz$
3. $xb + xc + yb + yc$
4. $xh + xk + yh + yk$
5. $xm + xn + my + ny$
6. $ah - ak + bh - bk$
7. $ax - ay + bx - by$
8. $am - bm + an - bn$
9. $hs + ht + ks + kt$
10. $xs - xt + ys - yt$
11. $ax - ay - bx + by$
12. $xs - xt - ys + yt$
13. $as - ay - xs + xy$
14. $hx - hy - bx + by$
15. $am - bm - an + bn$
16. $xk - xm - kz + mz$
17. $2ax + 6ay + bx + 3by$
18. $2ax + 2ay + bx + by$
19. $2mh - 2mk + nh - nk$
20. $2mh + 3mk - 2nh - 3nk$
21. $6ax + 2bx + 3ay + by$
22. $2ax - 2ay - bx + by$
23. $x^2a + x^2b + ya + yb$
24. $ms + 2mt^2 - ns - 2nt^2$

Example 1

Factorise $x^2 + 6x + 8$.

(a) Find two numbers which multiply to give 8 and add up to 6.
In this case the numbers are 4 and 2.

(b) Put these numbers into brackets.
So $x^2 + 6x + 8 = (x + 4)(x + 2)$

Example 2

Factorise (a) $x^2 + 2x - 15$
(b) $x^2 - 6x + 8$

(a) Two numbers which multiply to give -15 and add up to $+2$ are -3 and 5.

$\therefore \quad x^2 + 2x - 15 = (x - 3)(x + 5)$

(b) Two numbers which multiply to give $+8$ and add up to -6 are -2 and -4.

$\therefore \quad x^2 - 6x + 8 = (x - 2)(x - 4)$

Exercise 23

Factorise the following:

1. $x^2 + 7x + 10$
2. $x^2 + 7x + 12$
3. $x^2 + 8x + 15$
4. $x^2 + 10x + 21$
5. $x^2 + 8x + 12$
6. $y^2 + 12y + 35$
7. $y^2 + 11y + 24$
8. $y^2 + 10y + 25$
9. $y^2 + 15y + 36$
10. $a^2 - 3a - 10$
11. $a^2 - a - 12$
12. $z^2 + z - 6$
13. $x^2 - 2x - 35$
14. $x^2 - 5x - 24$
15. $x^2 - 6x + 8$
16. $y^2 - 5y + 6$
17. $x^2 - 8x + 15$
18. $a^2 - a - 6$
19. $a^2 + 14a + 45$
20. $b^2 - 4b - 21$
21. $x^2 - 8x + 16$
22. $y^2 + 2y + 1$
23. $y^2 - 3y - 28$
24. $x^2 - x - 20$
25. $x^2 - 8x - 240$
26. $x^2 - 26x + 165$
27. $y^2 + 3y - 108$
28. $x^2 - 49$
29. $x^2 - 9$
30. $x^2 - 16$

Example

Factorise $3x^2 + 13x + 4$.

(a) Find two numbers which multiply to give 12 and add up to 13.
In this case the numbers are 1 and 12.

(b) Split the '$13x$' term,
$3x^2 + x + 12x + 4$

(c) Factorise in pairs,
$x(3x + 1) + 4(3x + 1)$

(d) $(3x + 1)$ is common,
$(3x + 1)(x + 4)$

Exercise 24

Factorise the following:

1. $2x^2 + 5x + 3$
2. $2x^2 + 7x + 3$
3. $3x^2 + 7x + 2$
4. $2x^2 + 11x + 12$
5. $3x^2 + 8x + 4$
6. $2x^2 + 7x + 5$
7. $3x^2 - 5x - 2$
8. $2x^2 - x - 15$
9. $2x^2 + x - 21$
10. $3x^2 - 17x - 28$
11. $6x^2 + 7x + 2$
12. $12x^2 + 23x + 10$
13. $3x^2 - 11x + 6$
14. $3y^2 - 11y + 10$
15. $4y^2 - 23y + 15$
16. $6y^2 + 7y - 3$
17. $6x^2 - 27x + 30$
18. $10x^2 + 9x + 2$
19. $6x^2 - 19x + 3$
20. $8x^2 - 10x - 3$
21. $12x^2 + 4x - 5$
22. $16x^2 + 19x + 3$
23. $4a^2 - 4a + 1$
24. $12x^2 + 17x - 14$
25. $15x^2 + 44x - 3$
26. $48x^2 + 46x + 5$
27. $64y^2 + 4y - 3$
28. $120x^2 + 67x - 5$
29. $9x^2 - 1$
30. $4a^2 - 9$

The difference of two squares

$x^2 - y^2 = (x - y)(x + y)$
Remember this result.

Example

Factorise (a) $4a^2 - b^2$
(b) $3x^2 - 27y^2$

(a) $4a^2 - b^2 = (2a)^2 - b^2$
$= (2a - b)(2a + b)$

(b) $3x^2 - 27y^2 = 3(x^2 - 9y^2)$
$= 3[x^2 - (3y)^2]$
$= 3(x - 3y)(x + 3y)$

Exercise 25

Factorise the following:

1. $y^2 - a^2$
2. $m^2 - n^2$
3. $x^2 - t^2$
4. $y^2 - 1$
5. $x^2 - 9$
6. $a^2 - 25$
7. $x^2 - \frac{1}{4}$
8. $x^2 - \frac{1}{9}$

9. $4x^2 - y^2$ **10.** $a^2 - 4b^2$ **11.** $25x^2 - 4y^2$ **12.** $9x^2 - 16y^2$
13. $x^2 - \frac{y^2}{4}$ **14.** $9m^2 - \frac{4}{9}n^2$ **15.** $16t^2 - \frac{4}{25}s^2$ **16.** $4x^2 - \frac{z^2}{100}$
17. $x^3 - x$ **18.** $a^3 - ab^2$ **19.** $4x^3 - x$ **20.** $8x^3 - 2xy^2$
21. $12x^3 - 3xy^2$ **22.** $18m^3 - 8mn^2$ **23.** $5x^2 - 1\frac{1}{4}$ **24.** $50a^3 - 18ab^2$
25. $12x^2y - 3yz^2$ **26.** $36a^3b - 4ab^3$ **27.** $50a^5 - 8a^3b^2$ **28.** $36x^3y - 225xy^3$

Evaluate the following:

29. $81^2 - 80^2$ **30.** $102^2 - 100^2$ **31.** $225^2 - 215^2$ **32.** $1211^2 - 1210^2$
33. $723^2 - 720^2$ **34.** $3{\cdot}8^2 - 3{\cdot}7^2$ **35.** $5{\cdot}24^2 - 4{\cdot}76^2$ **36.** $1234^2 - 1235^2$
37. $3{\cdot}81^2 - 3{\cdot}8^2$ **38.** $540^2 - 550^2$ **39.** $7{\cdot}68^2 - 2{\cdot}32^2$ **40.** $0{\cdot}003^2 - 0{\cdot}002^2$

2.9 Quadratic equations

So far, we have met linear equations which have one solution only. Quadratic equations always have an x^2 term, and often an x term and a number term, and generally have two different solutions.

Solution by factors

Consider the equation $a \times b = 0$, where a and b are numbers. The product $a \times b$ can only be zero if either a or b (or both) is equal to zero. Can you think of other possible pairs of numbers which multiply together to give zero?

Example 1

Solve the equation $x^2 + x - 12 = 0$

Factorising, $(x - 3)(x + 4) = 0$
either $x - 3 = 0$ or $x + 4 = 0$
$x = 3$ $x = -4$

Example 2

Solve the equation $6x^2 + x - 2 = 0$

Factorising, $(2x - 1)(3x + 2) = 0$
either $2x - 1 = 0$ or $3x + 2 = 0$
$2x = 1$ $3x = -2$
$x = \frac{1}{2}$ $x = -\frac{2}{3}$

Exercise 26

Solve the following equations:

1. $x^2 + 7x + 12 = 0$ **2.** $x^2 + 7x + 10 = 0$ **3.** $x^2 + 2x - 15 = 0$
4. $x^2 + x - 6 = 0$ **5.** $x^2 - 8x + 12 = 0$ **6.** $x^2 + 10x + 21 = 0$
7. $x^2 - 5x + 6 = 0$ **8.** $x^2 - 4x - 5 = 0$ **9.** $x^2 + 5x - 14 = 0$
10. $2x^2 - 3x - 2 = 0$ **11.** $3x^2 + 10x - 8 = 0$ **12.** $2x^2 + 7x - 15 = 0$

13. $6x^2 - 13x + 6 = 0$
14. $4x^2 - 29x + 7 = 0$
15. $10x^2 - x - 3 = 0$
16. $y^2 - 15y + 56 = 0$
17. $12y^2 - 16y + 5 = 0$
18. $y^2 + 2y - 63 = 0$
19. $x^2 + 2x + 1 = 0$
20. $x^2 - 6x + 9 = 0$
21. $x^2 + 10x + 25 = 0$
22. $x^2 - 14x + 49 = 0$
23. $6a^2 - a - 1 = 0$
24. $4a^2 - 3a - 10 = 0$
25. $z^2 - 8z - 65 = 0$
26. $6x^2 + 17x - 3 = 0$
27. $10k^2 + 19k - 2 = 0$
28. $y^2 - 2y + 1 = 0$
29. $36x^2 + x - 2 = 0$
30. $20x^2 - 7x - 3 = 0$

Example 1

Solve the equation $x^2 - 7x = 0$

Factorising, $x(x-7) = 0$
either $x = 0$ or $x - 7 = 0$
$x = 7$

The solutions are $x = 0$ and $x = 7$.

Example 2

Solve the equation $4x^2 - 9 = 0$

(a) Factorising, $(2x-3)(2x+3) = 0$
either $2x - 3 = 0$ or $2x + 3 = 0$
$2x = 3$ $\quad 2x = -3$
$x = \frac{3}{2}$ $\quad x = -\frac{3}{2}$

(b) Alternative method
$4x^2 - 9 = 0$
$4x^2 = 9$
$x^2 = \frac{9}{4}$
$x = +\frac{3}{2}$ or $-\frac{3}{2}$.

Hint
You must give both the solutions. A common error is to only give the positive square root.

Exercise 27

Solve the following equations:

1. $x^2 - 3x = 0$
2. $x^2 + 7x = 0$
3. $2x^2 - 2x = 0$
4. $3x^2 - x = 0$
5. $x^2 - 16 = 0$
6. $x^2 - 49 = 0$
7. $4x^2 - 1 = 0$
8. $9x^2 - 4 = 0$
9. $6y^2 + 9y = 0$
10. $6a^2 - 9a = 0$
11. $10x^2 - 55x = 0$
12. $16x^2 - 1 = 0$
13. $y^2 - \frac{1}{4} = 0$
14. $56x^2 - 35x = 0$
15. $36x^2 - 3x = 0$
16. $x^2 = 6x$
17. $x^2 = 11x$
18. $2x^2 = 3x$
19. $x^2 = x$
20. $4x = x^2$
21. $3x - x^2 = 0$
22. $4x^2 = 1$
23. $9x^2 = 16$
24. $x^2 = 9$
25. $12x = 5x^2$
26. $1 - 9x^2 = 0$
27. $x^2 = \dfrac{x}{4}$
28. $2x^2 = \dfrac{x}{3}$
29. $4x^2 = \dfrac{1}{4}$
30. $\dfrac{x}{5} - x^2 = 0$

Solution by formula

The solutions of the quadratic equation $ax^2 + bx + c = 0$ are given by the formula

$$x = \frac{-b \pm \sqrt{(b^2 - 4ac)}}{2a}$$

Use this formula only after trying (and failing) to factorise.

Example

Solve the equation $2x^2 - 3x - 4 = 0$.
In this case $a = 2$, $b = -3$, $c = -4$.

$$x = \frac{-(-3) \pm \sqrt{[(-3)^2 - (4 \times 2 \times -4)]}}{2 \times 2}$$

$$x = \frac{3 \pm \sqrt{[9 + 32]}}{4} = \frac{3 \pm \sqrt{41}}{4} = \frac{3 \pm 6{\cdot}403}{4}$$

either $x = \frac{3 + 6{\cdot}403}{4}$ or $x = \frac{3 - 6{\cdot}403}{4} = \frac{-3{\cdot}403}{4}$

$= 2{\cdot}35$ (2 decimal places) $= -0{\cdot}85$ (2 decimal places).

Exercise 28

Solve the following, giving answers to two decimal places where necessary:

1. $2x^2 + 11x + 5 = 0$
2. $3x^2 + 11x + 6 = 0$
3. $6x^2 + 7x + 2 = 0$
4. $3x^2 - 10x + 3 = 0$
5. $5x^2 - 7x + 2 = 0$
6. $6x^2 - 11x + 3 = 0$
7. $2x^2 + 6x + 3 = 0$
8. $x^2 + 4x + 1 = 0$
9. $5x^2 - 5x + 1 = 0$
10. $x^2 - 7x + 2 = 0$
11. $2x^2 + 5x - 1 = 0$
12. $3x^2 + x - 3 = 0$
13. $3x^2 + 8x - 6 = 0$
14. $3x^2 - 7x - 20 = 0$
15. $2x^2 - 7x - 15 = 0$
16. $x^2 - 3x - 2 = 0$
17. $2x^2 + 6x - 1 = 0$
18. $6x^2 - 11x - 7 = 0$
19. $3x^2 + 25x + 8 = 0$
20. $3y^2 - 2y - 5 = 0$
21. $2y^2 - 5y + 1 = 0$
22. $\frac{1}{2}y^2 + 3y + 1 = 0$
23. $2 - x - 6x^2 = 0$
24. $3 + 4x - 2x^2 = 0$
25. $1 - 5x - 2x^2 = 0$
26. $3x^2 - 1 + 4x = 0$
27. $5x - x^2 + 2 = 0$
28. $24x^2 - 22x - 35 = 0$
29. $36x^2 - 17x - 35 = 0$
30. $20x^2 + 17x - 63 = 0$
31. $x^2 + 2{\cdot}5x - 6 = 0$
32. $0{\cdot}3y^2 + 0{\cdot}4y - 1{\cdot}5 = 0$
33. $10 - x - 3x^2 = 0$
34. $x^2 + 3{\cdot}3x - 0{\cdot}7 = 0$
35. $12 - 5x^2 - 11x = 0$
36. $5x - 2x^2 + 187 = 0$

The solution to a problem can involve an equation which does not at first appear to be quadratic. The terms in the equation may need to be rearranged as shown below.

Example

Solve:

$$2x(x - 1) = (x + 1)^2 - 5$$
$$2x^2 - 2x = x^2 + 2x + 1 - 5$$
$$2x^2 - 2x - x^2 - 2x - 1 + 5 = 0$$
$$x^2 - 4x + 4 = 0$$
$$(x - 2)(x - 2) = 0$$
$$x = 2$$

In this example the quadratic has a repeated solution of $x = 2$.

Exercise 29

Solve the following, giving answers to two decimal places where necessary:

1. $x^2 = 6 - x$

2. $x(x+10) = -21$

3. $3x + 2 = 2x^2$

4. $x^2 + 4 = 5x$

5. $6x(x+1) = 5 - x$

6. $(2x)^2 = x(x-14) - 5$

7. $(x-3)^2 = 10$

8. $(x+1)^2 - 10 = 2x(x-2)$

9. $(2x-1)^2 = (x-1)^2 + 8$

10. $3x(x+2) - x(x-2) + 6 = 0$

11. $x = \frac{15}{x} - 22$

12. $x + 5 = \frac{14}{x}$

13. $4x + \frac{7}{x} = 29$

14. $10x = 1 + \frac{3}{x}$

15. $2x^2 = 7x$

16. $16 = \frac{1}{x^2}$

17. $2x + 2 = \frac{7}{x} - 1$

18. $\frac{2}{x} + \frac{2}{x+1} = 3$

19. $\frac{3}{x-1} + \frac{3}{x+1} = 4$

20. $\frac{2}{x-2} + \frac{4}{x+1} = 3$

21. One of the solutions published by Cardan in 1545 for the solution of cubic equations is given below. For an equation in the form $x^3 + px = q$

$$x = \sqrt[3]{\left[\sqrt{\left(\frac{p}{3}\right)^3 + \left(\frac{q}{2}\right)^2} + \frac{q}{2}\right]} - \sqrt[3]{\left[\sqrt{\left(\frac{p}{3}\right)^3 + \left(\frac{q}{2}\right)^2} - \frac{q}{2}\right]}$$

Use the formula to solve the following equations, giving answers to 4 sig. fig. where necessary.

(a) $x^3 + 7x = -8$

(b) $x^3 + 6x = 4$

(c) $x^3 + 3x = 2$

(d) $x^3 + 9x - 2 = 0$

2.10 Problems solved by quadratic equations

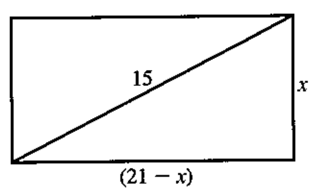
Example 1

The perimeter of a rectangle is 42 cm. If the diagonal is 15 cm, find the width of the rectangle.

Let the width of the rectangle be x cm.

Since the perimeter is 42 cm, the sum of the length and the width is 21 cm.

$\therefore$ length of rectangle $= (21 - x)$ cm



You can find out more about Pythagoras' Theorem in Unit 4 on page 119.

By Pythagoras' theorem

$$x^2 + (21 - x)^2 = 15^2$$
$$x^2 + (21 - x)(21 - x) = 15^2$$
$$x^2 + 441 - 42x + x^2 = 225$$
$$2x^2 - 42x + 216 = 0$$
$$x^2 - 21x + 108 = 0$$
$$(x - 12)(x - 9) = 0$$
$$x = 12$$
$$\text{or} \quad x = 9$$

Note that the dimensions of the rectangle are 9 cm and 12 cm, whichever value of x is taken.

$\therefore$ The width of the rectangle is 9 cm.

Example 2

A man bought a certain number of golf balls for $20. If each ball had cost 20 cents less, he could have bought five more for the same money. How many golf balls did he buy?

Let the number of balls bought be x.

Cost of each ball $= \dfrac{2000}{x}$ cents

If five more balls had been bought

Cost of each ball now $= \dfrac{2000}{(x+5)}$ cents

The new price is 20 cents less than the original price.

$$\therefore \quad \frac{2000}{x} - \frac{2000}{(x+5)} = 20$$

(multiply by x)

$$x \cdot \frac{2000}{x} - x \cdot \frac{2000}{(x+5)} = 20x$$

(multiply by $(x + 5)$)

$$2000(x+5) - x\frac{2000}{(x+5)}(x+5) = 20x(x+5)$$
$$2000x + 10\,000 - 2000x = 20x^2 + 100x$$
$$20x^2 + 100x - 10\,000 = 0$$
$$x^2 + 5x - 500 = 0$$
$$(x - 20)(x + 25) = 0$$
$$x = 20$$
$$\text{or} \quad x = -25$$

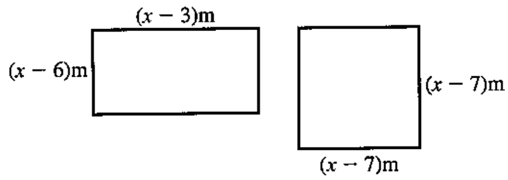
We discard $x = -25$ as meaningless.
The number of balls bought = 20.

Exercise 30

Solve by forming a quadratic equation:

Questions 4, 6 and 7 use Pythagoras' Theorem.

1. Two numbers, which differ by 3, have a product of 88. Find them.
2. The product of two consecutive odd numbers is 143. Find the numbers. (Hint: If the first odd number is x, what is the next odd number?)
3. The length of a rectangle exceeds the width by 7 cm. If the area is 60 cm^2, find the length of the rectangle.
4. The length of a rectangle exceeds the width by 2 cm. If the diagonal is 10 cm long, find the width of the rectangle.
5. The area of the rectangle exceeds the area of the square by 24 m^2. Find x.



6. The perimeter of a rectangle is 68 cm. If the diagonal is 26 cm, find the dimensions of the rectangle.
7. A man walks a certain distance due North and then the same distance plus a further 7 km due East. If the final distance from the starting point is 17 km, find the distances he walks North and East.
8. A farmer makes a profit of x cents on each of the $(x + 5)$ eggs her hen lays. If her total profit was 84 cents, find the number of eggs the hen lays.
9. A boy buys x eggs at $(x - 8)$ cents each and $(x - 2)$ rashers of bacon at $(x - 3)$ cents each. If the total bill is \$1·75, how many eggs does he buy?
10. A number exceeds four times its reciprocal by 3. Find the number.
11. Two numbers differ by 3. The sum of their reciprocals is $\frac{7}{10}$; find the numbers.
12. A cyclist travels 40 km at a speed of x km/h. Find the time taken in terms of x. Find the time taken when his speed is reduced by 2 km/h. If the difference between the times is 1 hour, find the original speed x.
13. An increase of speed of 4 km/h on a journey of 32 km reduces the time taken by 4 hours. Find the original speed.
14. A train normally travels 240 km at a certain speed. One day, due to bad weather, the train's speed is reduced by 20 km/h so that the journey takes two hours longer. Find the normal speed.

15. The speed of a sparrow is x km/h in still air. When the wind is blowing at 1 km/h, the sparrow takes 5 hours to fly 12 kilometres to her nest and 12 kilometres back again. She goes out directly into the wind and returns with the wind behind her. Find her speed in still air.

16. An aircraft flies a certain distance on a bearing of 135° and then twice the distance on a bearing of 225°. Its distance from the starting point is then 350 km. Find the length of the first part of the journey.

A bearing is a clockwise angle measured from North.

17. In Figure 1, ABCD is a rectangle with AB = 12 cm and BC = 7 cm. AK = BL = CM = DN = x cm. If the area of KLMN is 54 cm^2 find x.

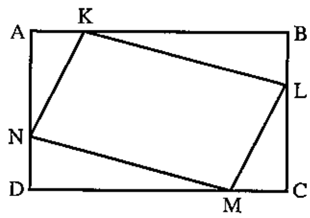


Figure 1

18. In Figure 1, AB = 14 cm, BC = 11 cm and AK = BL = CM = DN = x cm. If the area of KLMN is now 97 cm^2, find x.

19. The numerator of a fraction is 1 less than the denominator. When both numerator and denominator are increased by 2, the fraction is increased by $\frac{1}{12}$. Find the original fraction.

20. The perimeters of a square and a rectangle are equal. One side of the rectangle is 11 cm and the area of the square is 4 cm^2 more than the area of the rectangle. Find the side of the square.

Revision exercise 2A

Questions 13 and 20 use Pythagoras' Theorem.

1. Solve the equations:
(a) $x + 4 = 3x + 9$ (b) $9 - 3a = 1$
(c) $y^2 + 5y = 0$ (d) $x^2 - 4 = 0$
(e) $3x^2 + 7x - 40 = 0$

2. Given $a = 3$, $b = 4$ and $c = -2$, evaluate:
(a) $2a^2 - b$ (b) $a(b - c)$
(c) $2b^2 - c^2$

3. Factorise completely:
(a) $4x^2 - y^2$ (b) $2x^2 + 8x + 6$
(c) $6m + 4n - 9km - 6kn$
(d) $2x^2 - 5x - 3$

4. Solve the simultaneous equations:
(a) $3x + 2y = 5$, $2x - y = 8$
(b) $2m - n = 6$, $2m + 3n = -6$
(c) $3x - 4y = 19$, $x + 6y = 10$
(d) $3x - 7y = 11$, $2x - 3y = 4$

5. Given that $x = 4$, $y = 3$, $z = -2$, evaluate:
(a) $2x(y + z)$ (b) $(xy)^2 - z^2$
(c) $x^2 + y^2 + z^2$ (d) $(x + y)(x - z)$
(e) $\sqrt{[x(1 - 4z)]}$ (f) $\frac{xy}{z}$

6. (a) Simplify $3(2x - 5) - 2(2x + 3)$.
(b) Factorise $2a - 3b - 4xa + 6xb$.
(c) Solve the equation $\frac{x - 11}{2} - \frac{x - 3}{5} = 2$.

7. Solve the equations:
(a) $5 - 7x = 4 - 6x$ (b) $\frac{7}{x} = \frac{2}{3}$
(c) $2x^2 - 7x = 0$ (d) $x^2 + 5x + 6 = 0$
(e) $\frac{1}{x} + \frac{1}{4} = \frac{1}{3}$

8. Factorise completely:
(a) $z^3 - 16z$ (b) $x^2y^2 + x^2 + y^2 + 1$
(c) $2x^2 + 11x + 12$

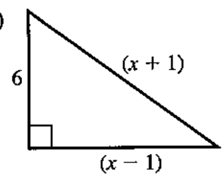
9. Find the value of $\frac{2x - 3y}{5x + 2y}$ when $x = 2a$ and $y = -a$.

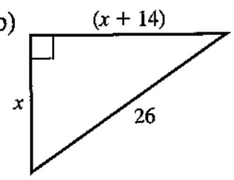
10. Solve the simultaneous equations:
(a) $7c + 3d = 29$
$5c - 4d = 33$
(b) $2x - 3y = 7$
$2y - 3x = -8$
(c) $5x = 3(1 - y)$
$3x + 2y + 1 = 0$
(d) $5s + 3t = 16$
$11s + 7t = 34$

11. Solve the equations:
(a) $4(y + 1) = \frac{3}{1 - y}$
(b) $4(2x - 1) - 3(1 - x) = 0$
(c) $\frac{x + 3}{x} = 2$
(d) $x^2 = 5x$

12. Solve the following, giving your answers correct to two decimal places.
(a) $2x^2 - 3x - 1 = 0$ (b) $x^2 - x - 1 = 0$
(c) $3x^2 + 2x - 4 = 0$ (d) $x + 3 = \frac{7}{x}$

13. Find x by forming a suitable equation.

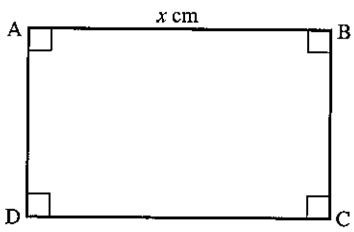
(a)


(b)


14. Given that $m = -2$, $n = 4$, evaluate:
(a) $5m + 3n$ (b) $5 + 2m - m^2$
(c) $m^2 + 2n^2$ (d) $(2m + n)(2m - n)$
(e) $(n - m)^2$ (f) $n - mn - 2m^2$

15. A car travels for x hours at a speed of $(x + 2)$ km/h. If the distance travelled is 15 km, write down an equation for x and solve it to find the speed of the car.

16. ABCD is a rectangle, where $AB = x$ cm and BC is 1·5 cm less than AB.



If the area of the rectangle is 52 cm², form an equation in x and solve it to find the dimensions of the rectangle.

17. Solve the equations:
(a) $(2x + 1)^2 = (x + 5)^2$
(b) $\frac{x+2}{2} - \frac{x-1}{3} = \frac{x}{4}$
(c) $x^2 - 7x + 5 = 0$, giving the answers correct to two decimal places.

18. Solve the equation:
$$\frac{x}{x+1} - \frac{x+1}{3x-1} = \frac{1}{4}$$

19. Given that $a + b = 2$ and that $a^2 + b^2 = 6$, prove that $2ab = -2$. Find also the value of $(a - b)^2$.

20. The sides of a right-angled triangle have lengths $(x - 3)$ cm, $(x + 11)$ cm and $2x$ cm, where $2x$ is the hypotenuse. Find x.

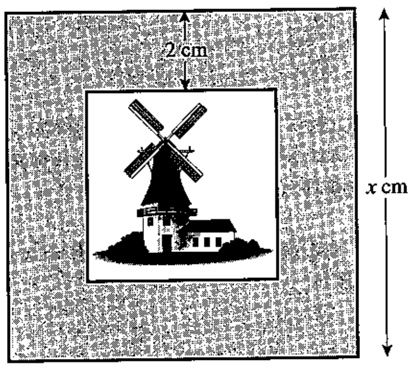
21. A piggy-bank contains 50 coins, all either 2 cents or 5 cents. The total value of the coins is \$1·87. How many 2 cents coins are there?

22. Pat bought 45 stamps, some for 10c and some for 18c. If he spent \$6·66 altogether, how many 10c stamps did he buy?

23. When each edge of a cube is decreased by 1 cm, its volume is decreased by 91 cm³. Find the length of a side of the original cube.

24. One solution of the equation $2x^2 - 7x + k = 0$ is $x = -\frac{1}{2}$. Find the value of k.

Examination exercise 2B



1. The diagram shows a square picture in a square frame of side x cm.
The width of the border all round the picture is 2 cm, and the area of the border is 112 cm^2.
(a) Use this information to form an equation in x.
(b) Solve your equation to find the value of x. N 96 2

2. A bank uses the formula $A = P\left(1 + \frac{r}{100}\right)^n$ to calculate the amount of money in an account.
(a) Calculate A when $P = 800$, $r = 6$ and $n = 5$, correct to two decimal places.
(b) When $n = 1$, the formula is
$$A = P\left(1 + \frac{r}{100}\right).$$
Make r the subject of this formula. N 97 2

3. In a set of three numbers,
the first is a positive integer,
the second is three more than the first
and the third is the square of the second.
(a) The first number in the set is x.
Write the second and third numbers in terms of x.
(b) The sum of the three numbers is 77.
(i) Write down an equation in x.
(ii) Show that your equation simplifies to $x^2 + 8x - 65 = 0$.
(iii) Solve the equation $x^2 + 8x - 65 = 0$.
(iv) Write down the three numbers. N 96 4

4. **Give exact answers to each part of this question.**
It is given that
$$1^2 + 2^2 + 3^2 + \ldots + k^2 = \frac{k(k+1)(2k+1)}{6}.$$
(a) Substitute $k = 100$ in the formula above to find the value of $1^2 + 2^2 + 3^2 + \ldots + 100^2$.
(b) $2^2 + 4^2 + 6^2 + \ldots + 100^2 = 2^2(1^2 + 2^2 + 3^2 + \ldots + n^2)$.
(i) Write down the value of n.
(ii) Hence find the value of $2^2 + 4^2 + 6^2 + \ldots + 100^2$.
(c) Use your answers to parts (a) and (b)(ii) to find the value of $1^2 + 3^2 + 5^2 + \ldots + 99^2$.
(d) Use some of your previous answers to find the value of:
$1^2 - 2^2 + 3^2 - 4^2 + 5^2 - 6^2 + \ldots + 99^2 - 100^2$ J 95 4

5. (a) Write as a single fraction:

$$\frac{2x+1}{3} - \frac{x-1}{2}$$

(b) (i) Factorise: $x^2 - 5x + 6$

(ii) Simplify:

$$\frac{x^2 - 5x + 6}{x^2 + x - 6}$$

(c) Solve the equation:

$3x^2 = 7x - 1$

Show all your working and give your answers correct to two decimal places. N 96 4

6. (a) (i) Write down the next two terms in the sequence:

$$\frac{1}{2}, \frac{2}{3}, \frac{3}{5}, \frac{5}{8}, \frac{8}{13}, \_\_, \_\_$$

(ii) This can be written in the form:

$$\frac{a}{b}, \frac{b}{a+b}, \frac{a+b}{a+2b}, \_\_\_, \_\_\_$$

Write down the next two terms of the sequence in terms of a and b.

(b) A different sequence follows the pattern:

$$\frac{1}{x}, \frac{2}{x+1}, \frac{3}{x+2}, \frac{4}{x+3}, \_\_\_, \_\_\_$$

(i) Write down the next two terms of this sequence.

(ii) Write down the 100th term of this sequence.

(iii) Find x if the *tenth* term equals $\frac{1}{2}$. J 96 4

7. Three positive integers are $(x - 1)$, x and $(x + 1)$.
When they are multiplied together the answer is 40 times their sum.

(a) (i) Write down an equation in x.

(ii) Show that your equation simplifies to $x^3 - 121x = 0$.

(b) Factorise **completely**, $x^3 - 121x$.

(c) Find the three positive integers. J 98 2

8. (a) Write the expression $\frac{100}{x-2} - \frac{100}{x}$ as a single fraction and simplify your answer.

(b) Rice costs x francs for one kilogram. How many kilograms can I buy for 100 francs?

(c) When rice costs $(x - 2)$ francs for one kilogram, I can buy five more kilograms for 100 francs. Write down an equation in x. Show that it simplifies to $x^2 - 2x - 40 = 0$.

(d) (i) Solve the equation $x^2 - 2x - 40 = 0$, giving your answers correct to two decimal places. Show all your working.

(ii) Write down the original price of one kilogram of rice.

J 98 4